Deductive Logic

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Hackett Publishing Company, Inc.
Indianapolis/Cambridge

PART I

TRUTH-FUNCTIONAL LOGIC

A. Analysis

§1. Statements

Truth-functional logic concerns several ways in which statements may be compounded to form more complex statements. These compounding methods typically use the connectives "and", "not", "or", and "if ... then", as in the following examples:

> Gladstone approved or Disraeli abstained.
> If Gladstone approved then Disraeli abstained.
> Gladstone approved and Disraeli did not abstain.

Our aim is to analyze such compounds in a systematic manner. We seek to formulate laws that tell us how the truth of a compound statement depends upon the truth of its simpler constituent statements. These laws will yield, for example, that the third statement above is true just in case the constituent statement "Gladstone approved" is true and the constituent statement "Disraeli abstained" is false. Moreover, these laws will give us the means to delineate the interdependencies among compound statements. For example, the second and third statements above cannot both be true. In such interdependencies logical argumentation is grounded.

The statements of which we speak, in this part of logic, are a particular kind of sentence. Let us clarify this notion. A statement is a sentence that is true or is false. Thus all statements are declarative sentences, since nondeclarative sentences—like "Where is Moose Jaw?", "O, to be in England!", and "Please pass the salt."—are neither true nor false. However, strictly construed, even many declarative sentences fail to be statements, for we require that statements be true or false, once and for all. The sentence "I am myopic" is, intrinsically, neither true nor false: it may be true when uttered by one speaker and false when uttered by another. Similarly, "She is British" uttered in some contexts may be true while uttered in other contexts may be false; the person "she" refers to varies with the context. The sentence "Seattle is far from here" is true in Philadelphia and false in Vancouver. Sentences containing "I", "she", or "here" are *context-dependent:* their truth or falsity is not fixed independently of the context of utterance. To obtain a statement from such a sentence, these and similar words must be replaced by words or phrases not subject to the influence of context. For example, the sentence just mentioned would be replaced by "Seattle is far from Philadelphia" or by "Seattle is far from Vancouver", depending upon where it is uttered.

Even sentences like "Roosevelt is impatient" are in some measure context-dependent; a conversational setting is needed to determine whether "Roosevelt" refers to, say, Theodore rather than to Franklin Delano. This dependence may be eliminated by expanding the sentence to "President Franklin Delano Roosevelt is impatient". But this sentence is still context-dependent, insofar as it may be true at one time and false at another, varying with FDR's temperamental state. Similarly,

Maria Callas sang at the Metropolitan Opera

was false before 1947 and is now true; and

Maria Callas will sing at the Metropolitan Opera

was true in 1950 and is now, alas, false. To eliminate this dependence on time of utterance, explicit mention of the time must be added. Uttered on May 6, 1963, these sentences correspond to the statements "Maria Callas sings at the Metropolitan Opera before May 6, 1963" and "Maria Callas sings at the Metropolitan Opera after May 6, 1963". Note that once the time is made explicit, the tense of the verb no longer matters; the verb may be said to be used *tenselessly.*

In sum, a statement is a sentence that is determinately true or determinately false independently of the circumstances in which it is used, independently of speaker, audience, time, place, and conversational context. Thus most sentences encountered in ordinary discourse are not, as they stand, statements in the strict sense.

This strict notion of a statement is important as a theoretical basis of our analysis. By paraphrasing the sentences of everyday discourse by statements, we give explicit recognition, for example, to the fact that you do not contradict me if you respond to my assertion "I'm hungry" by saying "I'm not", whereas you do contradict me if you respond to my claiming "Montpelier is the capital of Vermont" by saying "Montpelier is not the capital of Vermont". In theory, then, the first step toward the logical analysis of a stretch of discourse is to paraphrase each sentence under consideration by a statement. In practice we avoid this tedium by imagining the sentences of our examples to be paraphrasable into statements *uniformly.* That is to say, we imagine them to have been uttered by a single speaker, at a single time, in a conversational setting that uniformly resolves any ambiguities. This tacit assumption enables us to treat sentences like "Gladstone approved", "Roosevelt is impatient", and "I am myopic", when they appear in our examples, as though they were statements.

§2. Conjunction

If I assert the statement

(1) Zola was a novelist and Rimbaud was a poet,

then I commit myself to both of the statements

(2) Zola was a novelist

(3) Rimbaud was a poet.

Moreover, if I assert each of (2) and (3), I must acquiesce in (1). For statement (1) is true if both (2) and (3) are true, and is false otherwise. Statement (1) is a compound statement called the *conjunction* of statement (2) and statement (3).

The conjunction of two statements is true if both of the two statements are true, and is false if at least one of the two statements is false.

In fact, statement (1) is true; but the following three conjunctions are false:

Zola was a novelist and Manet was a poet

2 + 3 = 6 and 3 + 7 = 10

Calgary is in Manitoba and Moose Jaw is in Ontario.

For in each case at least one of the conjoined statements—or *conjuncts*, as they are called—is false.

As these examples suggest, conjunctions can be constructed by connecting the two conjuncts with the word "and". In our logical symbolism we use a dot "." for conjunction. Thus, statement (1) may be written:

Zola was a novelist . Rimbaud was a poet

and if we use "p" and "q" to represent statements, their conjunction is written "$p.q$".

Any two statements may be conjoined. In particular, conjunction may be iterated. A conjunction "$p.q$" may be conjoined with "r", and the result written "$(p.q).r$". The parentheses here serve to group the first conjunct of this conjunction, namely "$p.q$", as a single whole. Now "$(p.q).r$" is true if and only if both "$p.q$" is true and "r" is true; and "$p.q$" is true if and only if both "p" and "q" are true. So "$(p.q).r$" is true if and only if all of "p", "q", and "r" are true. Similarly, we may conjoin "p" with the conjunction "$q.r$", obtaining "$p.(q.r)$". The latter is true if and only if "p" and "$q.r$" are both true; and "$q.r$" is true if and only if "q" and "r" are both true. So "$p.(q.r)$" is true if and only if all of "p", "q", and "r" are true.

Thus there is no difference between "$(p.q).r$" and "$p.(q.r)$". In a word, conjunction is *associative:* the internal grouping in an iterated conjunction doesn't matter. Consequently, we may write "$p.q.r$", without parentheses. This compound, which we may call the conjunction of "p", "q", and "r", may be construed as "$(p.q).r$" or as "$p.(q.r)$"; either way, it is true just in case all of "p", "q", and "r" are true.

In being associative, conjunction is like addition and multiplication in arithmetic. That is, $(x + y) + z = x + (y + z)$ and $(x \times y) \times z = x \times (y \times z)$ for all numbers x, y, and z. Hence we can, and do, write iterated sums like $x + y + z$ and iterated products like $x \times y \times z$ without parentheses. (Division, on the other hand, is not associative. $(24 \div 4) \div 2$ is 3, whereas $24 \div (4 \div 2)$ is 12. The expression "$24 \div 4 \div 2$" is ambiguous and cannot be used; parentheses are essential.)

Clearly, we may conjoin any number of statements: "$p.q.r.s.t$" represents a conjunction of five statements; it is true if its conjuncts are all true, and is false otherwise. Iterated conjunctions can be expressed in English by joining the conjuncts with commas and inserting "and" just before the last conjunct. "Zola was a novelist, Rimbaud was a poet, Manet was a painter, and Rodin was a sculptor" is a conjunction with four conjuncts.

Aside from being associative, conjunction is also *commutative:* the order of the conjuncts makes no difference. That is, "$p.q$" and "$q.p$" amount to the same, for they are true if "p" and "q" are both true and they are false otherwise. Here again, conjunction is like addition and multiplication.

We have seen that the conjunction of two statements may be expressed by connecting the two conjuncts with "and". Now, "and" is used in English not just between statements but also to connect nouns, verbs, adverbs, and other parts of speech. Statements containing "and" in these ways may ordinarily be analyzed as conjunctions. Thus the statements

Fred sang and danced

Putin quoted Kant and Hegel

Agassi volleyed confidently and effortlessly

may be identified with the conjunctions

Fred sang . Fred danced

Putin quoted Kant . Putin quoted Hegel

Agassi volleyed confidently . Agassi volleyed effortlessly.

In the analysis of a statement as a conjunction, care must be taken with pronouns. For example,

Scrooge gave Cratchit food and paid his bills

cannot be analyzed as

Scrooge gave Cratchit food . Scrooge paid his bills.

For in the statement "Scrooge paid his bills" the pronoun "his" refers to Scrooge, whereas in the original statement "his" refers to Cratchit. Thus the correct analysis is

Scrooge gave Cratchit food . Scrooge paid Cratchit's bills.

On the other hand, the statement

Scrooge gave Cratchit food and regretted his previous parsimony

can correctly be rendered

Scrooge gave Cratchit food . Scrooge regretted his previous parsimony.

Pronouns, in sum, require attention: in many cases their antecedents have to be supplied in the process of analysis, although in some cases they do not.

The general rule that "and" expresses conjunction has exceptions. The statement

(4) Callas walked out and the audience booed

on its most natural interpretation conveys more than a conjunction. It conveys a succession in time, so that its truth does not depend on merely the truth, separately, of the two statements "Callas walked out" and "the audience booed". Thus the statement is not a conjunction. Another way to see this is to contrast (4) with

(5) The audience booed and Callas walked out.

Were (4) and (5) conjunctions, they would amount to the same, since conjunction is commutative. But in fact (5) conveys a picture of events rather different from (4).

Exceptions occur also when "and" is used between parts of speech. To take the statement "Eggers wrote a bestseller and became wealthy" as a conjunction fails to do it justice; as in the previous example, this statement would ordinarily be understood as conveying a succession in time. Exceptional for a different reason is

Fred and Ginger danced the night away.

This ought not be identified with

Fred danced the night away . Ginger danced the night away,

since it conveys more than that each of Fred and Ginger danced nightlong, but that they did it together. We might say that "and" here expresses not conjunction but rather a "collective subject". So too in "June, July, and August make up the summer recess" and "Brutus and Cassius conspired". An ambiguous case is

New York is bigger than Boston and Philadelphia.

This statement is a conjunction if meant to say that New York is bigger than each, but is not a conjunction if meant to say that New York is bigger than the two put together.

Nonconjunctive uses of "and" are exceptions. In the great majority of cases "and" does express conjunction. Whether any given occurrence of "and" does or does not depends upon what the statement is supposed to convey. There are no general rules for deciding this; you must rely on your ability to understand English, and your knowledge of the circumstances in which the statement is uttered. Of course, sometimes the answer will not be evident, even if circumstances of utterance are taken into account. But that is just to say that sometimes people speak ambiguously.

§3. Negation

To assert the statement

(1) Zola was not a novelist

is just to deny the statement

(2) Zola was a novelist.

Statement (1) is called the *negation* of statement (2).

The negation of a statement is true if the negated statement is false, and is false if the negated statement is true.

We ordinarily express the negation of simple sentence by using the word "not" (or "does not" and its conjugations), as in (1), or in "85 is not a prime number", or in "Zola did not write poetry". This rule often does not hold for more complex statements. The negation of "Both President Lyndon Johnson and Mrs. Johnson were from Texas" is not "Both President Lyndon Johnson and Mrs. Johnson were not from Texas", but rather "Either President Lyndon Johnson or Mrs. Johnson, or both, were not from Texas". The negation of "It sometimes rains in Seattle" is not "It sometimes does not rain in Seattle" but rather "It never rains in Seattle". Indeed, "It sometimes does not rain in Seattle" is the negation of "It always rains in Seattle". (Negations of complex statements will be examined in more detail in §5 and §20 below.) Luckily, in almost every case the negation of a statement can be expressed by prefixing the statement with "it is not the case that". Thus we have the long-winded but serviceable "It is not the case that it sometimes rains in Seattle". To avoid the vagaries of ordinary language, in our logical notation we symbolize negation by a bar "–". The statements

–(Zola was a novelist)

–(Zola wrote poetry)

–(It sometimes rains in Seattle)

are the negations of the statements within the parentheses. When a statement is represented as a single letter, like "*p*", or is itself a negation, like "$-(p.q)$", parentheses may be dropped. Thus "$-p$" is the negation of "*p*" and "$--(p.q)$" is the negation of "$-(p.q)$".

It should be clear that "$--p$" amounts to the same thing as "*p*". For "$--p$" is true just in case "$-p$" is false, and "$-p$" is false just in case "*p*" is true. Double negations, therefore, are redundant.

§4. Disjunction

If I assert the statement

(1) Callas gave an uninspired performance or the audience was predisposed against her

then I must agree that at least one of the statements

(2) Callas gave an uninspired performance

(3) The audience was predisposed against Callas

is true, even though I might not know which. Moreover, I do not preclude the possibility that both (2) and (3) are true; I am denying only that (2) and (3) are both false. Statement (1) is the *disjunction* of (2) and (3); it is true if at least one of (2) and (3) is true, and is false otherwise. Statements (2) and (3) are the *disjuncts* of (1).

This account of the logical behavior of (1) might perhaps be met with some hesitancy. The hesitancy arises because "or" has two precise but conflicting senses in English. The sense just ascribed to the "or" in (1) is called the *inclusive* sense. The contrasting *exclusive* sense is that under which "*p* or *q*" counts as true if and only if exactly one of "*p*" and "*q*" is true. Inclusive "or" and exclusive "or" differ only in the case that both constituent statements are true; in this case "*p* or *q*" is true when "or" is inclusive, and is false when "or" is exclusive.

To find an instance in which "or" must be interpreted exclusively, we must provide circumstances in which a person is using the statement containing "or" explicitly to deny that both constituent statements are true. Here is a well-worn example. Suppose a child is pleading to be taken both to the beach and to the movies, and the parent replies:

> We will go to the beach or we will go to the movies.

The exclusive nature of "or" here is clear: the parent is promising one outing but precluding both.

More common are instances in which "or" should be interpreted inclusively. It seems to me that (1) is such an instance. (Think, for example, of (1) as uttered to explain a hostile reception to a Callas performance. Surely one would not want to call this explanation false if both (2) and (3) were true.) Similarly, suppose the rule-book says that a student satisfies the logic requirement on the condition that

> The student takes the Deductive Logic course or the student passes the departmental examination.

If the student overzealously does both, then clearly the condition would still be considered true, and the student taken to have satisfied the requirement. Thus "or" is inclusive here.

Sometimes it does not matter which sense is assigned to an occurrence of "or", in that either would do equally well. For example, in

> The Prime Minister is in London or Ottawa,

which is a condensed form of

> The Prime Minister is in London or the Prime Minister is in Ottawa,

it makes no difference which way "or" is construed. A difference could arise only when both constituent statements are true, but—since the Prime Minister cannot be in two places at once—this situation never arises. Thus the speaker of this sentence need have no concern with such a situation.

The decision as to which sense, exclusive or inclusive, ought be assigned an occurrence of "or" depends upon what is supposed to be conveyed. If there is a significant danger of confusion, we might use a more elaborate form of words to pin down the appropriate sense: for the exclusive sense "p or q but not both"; for the inclusive sense "p or q or both" (or even the horribly inelegant "p and/or q").

On the whole, though, inclusive "or" seems to be more common than exclusive "or", while many instances are of the don't-care variety mentioned two paragraphs above. Hence we do little injustice to everyday language if we interpret "or" inclusively in all cases but those few that are glaringly exclusive. All uses of "or" below should be construed inclusively. We reserve the term "disjunction" for inclusive "or", and in our logical symbolism represent it by the wedge "$\vee$". Thus "$p \vee q$" represents the disjunction of "p" and "q". (The symbol "$\vee$" comes from the "v" in "vel", the Latin for inclusive "or". Latin had another word, "aut", for exclusive "or". I know no modern language in which the distinction is preserved.) To repeat, then:

The disjunction of two statements is true if at least one of those statements is true, and is false if neither of those statements is true.

By the way, our decision to give disjunction a place in our symbolism and to make no special provision for exclusive "or" does not preclude us from expressing the latter symbolically. Indeed, "p or q but not both" may be represented as "$(p \vee q) \, . \, -(p.q)$", or, alternatively, as "$(p.-q) \vee (-p.q)$". The reader should check that these are in fact correct notations.

It should be clear, upon reflection, that disjunction is associative: "$(p \vee q) \vee r$" and "$p \vee (q \vee r)$" amount to the same, since each is true if at least one of "p", "q", and "r" is true, and each is false if "p", "q", and "r" are all false. Thus we may write such disjunctions without parentheses, for example "$p \vee q \vee r$" and "$p \vee q \vee r \vee s$". Disjunction is also commutative: there is no difference between "$p \vee q$" and "$q \vee p$".

We have been talking of "or" as used to connect statements, but "or" may also occur between parts of speech. As with "and", statements that contain "or" in such ways can usually be taken to be condensed forms of statements in which "or" connects statements, and hence be analyzed as disjunctions of those statements.

§5. Grouping

The associativity of conjunction and disjunction allows us to ignore internal grouping in iterated conjunctions and iterated disjunctions. But grouping is essential in compound statements that involve conjunction, negation, and disjunction in combination. Consider

> Figaro exulted, and Basilio fretted, or the Count had a plan.

This is ambiguous as it stands; it could be expressing either of the following:

(1) (Figaro exulted . Basilio fretted) ∨ the Count had a plan

(2) Figaro exulted . (Basilio fretted ∨ the Count had a plan).

The distinction we are drawing is that between "$(p.q) \vee r$" and "$p.(q \vee r)$". These two compounds behave very differently. If, for example, "p" is false, "q" is false, and "r" is true, then the former is true, since it is a disjunction whose second disjunct is true, while the latter is false, since it is a conjunction whose first conjunct is false. Thus we must insist on grouping. In English various expedients are available for this: the use of "either ... or" rather than simply "or", where the placement of "either" serves to identify the extent of the first disjunct; the use of emphatic particles like "else", which makes an "or" mark a stronger break, and like "moreover", which makes an "and" mark a stronger break; and various types of punctuation. Thus (1) might be expressed in these two ways:

> Either Figaro exulted and Basilio fretted or the Count had a plan
>
> Figaro exulted and Basilio fretted, or else the Count had a plan,

and (2) might be expressed in these three:

> Figaro exulted and either Basilio fretted or the Count had a plan
>
> Figaro exulted and, moreover, Basilio fretted or the Count had a plan
>
> Figaro exulted; and Basilio fretted or the Count had a plan.

Grouping can sometimes be enforced by condensation, that is, by using "and" or "or" between parts of speech rather than between statements. Thus the statements

Fred danced and sang or Ginger sang
Fred danced and Fred or Ginger sang

are unambiguously of the forms "$(p.q) \vee r$" and "$p.(q \vee r)$", respectively.

Grouping is also essential in combinations of conjunction or disjunction with negation. We must distinguish "$-p.q$" from "$-(p.q)$", and "$-p \vee q$" from "$-(p \vee q)$": in the first of each pair only "p" is negated, while in the second the whole compound is negated. We must also distinguish between $-(p.q)$" and "$-p.-q$", and between "$-(p \vee q)$" and "$-p \vee -q$". Consider the four possible cases:

"p" and "q" both true
"p" true, "q" false
"p" false, "q" true
"p" and "q" both false.

Now "$-(p.q)$" is the negation of "$p.q$", and so is false in the first case and true in the others; whereas "$-p.-q$" is true in the fourth case only. And "$-p \vee -q$" is true when one or both of "$-p$" and "$-q$" is true, that is, in all but the first case; while "$-(p \vee q)$" is true when "$p \vee q$" fails, that is, in the fourth case only.

We see then that "$-(p.q)$" agrees with "$-p \vee -q$", and that "$-(p \vee q)$" agrees with "$-p.-q$". The negation of a conjunction amounts to a disjunction of negations, and the negation of a disjunction to a conjunction of negations. (These equivalences are called *De Morgan's Laws.*) Note that "$-p.-q$" may be rendered in ordinary language as "Neither p nor q", and so it should occasion no surprise that it amounts to "Not: either p or q", that is, to "$-(p \vee q)$".

§6. Truth-functions

To determine whether "$-p$", "$p.q$", and "$p \vee q$" are true, one needs to know only whether "p" is true and whether "q" is true. We call truth and falsity *truth-values*, and we say that the truth-value of a statement is truth or falsity according to whether the statement is true or false. Thus we may say that the truth-value of a negation, conjunction, or disjunction depends only on the truth-values of its constituent statements. For this reason we call negation, conjunction, and disjunction *truth-functions.*

Some ways of compounding statements to form more complex statements are not truth-functional. For example, statements may be compounded with "because"

(1) The Confederacy was defeated because Britain did not recognize it.

Now, we all agree that the Confederacy was defeated and that Britain did not recognize the Confederacy, yet we still might disagree about the truth of (1). Thus the truth-value of (1) does not depend solely on the truth-values of its constituent statements. Indeed, a constituent of (1) may be replaced with another statement that has the same truth-value, and the truth-value of the whole will be altered. For example, by one such replacement we can obtain on the one hand the obviously false "The Confederacy was defeated because General Lee had a beard", and by another such replacement we can obtain the obviously true "There was no British Embassy in Richmond because Britain did not recognize the Confederacy". The truth-values of truth-functional compounds, in contrast, are never affected when a constituent statement is replaced by another statement of like truth-value.

Truth-functions are completely characterized by the rules that tell *how* the truth-value of the whole is determined by the truth-values of the constituents. We may give a convenient graphic representation of these rules by means of what are called *truth-tables.* This is the truth-table for conjunction:

p	q	$p.q$
⊤	⊤	⊤
⊤	⊥	⊥
⊥	⊤	⊥
⊥	⊥	⊥

Here "⊤" represents truth and "⊥" falsity. The four lines of the truth-table represent the four possible cases: "p" and "q" both true, "p" true and "q" false, "p" false and "q" true, "p" and "q" both false. The entries in the last column then tell us that "$p.q$" is true in the first of these cases and false in the other three.

These are the truth-tables for disjunction and negation:

p	q	$p \vee q$
⊤	⊤	⊤
⊤	⊥	⊤
⊥	⊤	⊤
⊥	⊥	⊥

p	$-p$
⊤	⊥
⊥	⊤

Since negation, conjunction, and disjunction are truth-functions, anything obtained by repeatedly combining these connectives will also be a truth-functional compound of its constituents. Hence the behavior of any such compound can

be completely exhibited by truth-tables. Here, for example, is the truth-table for "$-(p.q)$":

p	q	$-(p.q)$
⊤	⊤	⊥
⊤	⊥	⊤
⊥	⊤	⊤
⊥	⊥	⊤

That is, as we have seen, "$-(p.q)$" is true if at least one of "p" and "q" is false, and is false if both "p" and "q" are true.

Of course, truth-functional compounds far more complex than "$-(p.q)$" can be formed by using negation, conjunction, and disjunction in combination. In §9 we give a procedure for constructing the truth-table for any such compound.

§7. Conditional

A statement of the form "if p then q" is called a *conditional;* the statement in the "p"-position is called the *antecedent* of the conditional, and that in the "q"-position the *consequent.* We seek a truth-functional connective that does justice, or at least reasonably good justice, to the use of conditional statements.

Now, if "p" is true then surely "if p then q" stands or falls on the truth of "q". If I say

(1) If today is Tuesday then we are in Paris,

and the day is Tuesday, then I have said something true if we are in Paris and something false if we are not. Thus the first two lines of the truth-table for "if p then q" should look like this:

p	q	if p then q
⊤	⊤	⊤
⊤	⊥	⊥

What, then, of the remaining two lines, those on which "p" is false? This question is somewhat artificial. In common practice, if someone asserts a statement of the form "if p then q" and the antecedent turns out to be false, the assertion is simply ignored, and the question of its truth or falsity is just not considered. In a sense, we ordinarily do not treat utterances of the form "if p then q" as statements, that is, as utterances which may always be assessed for truth-values as wholes. Our decision as logicians to treat conditionals as statements is thus something of a departure from everyday attitudes, although hardly a serious one, and one that is essential to the logical analysis of complex compounds.

Moreover, one aspect of our common practice does suggest a suitable completion of the truth-table. If I assert a conditional whose antecedent turns out false, I certainly would not be charged with having uttered a falsehood. So let us take the conditional to be true in such cases. That is, let us adopt the following as the truth-table for conditional:

p	q	if p then q
⊤	⊤	⊤
⊤	⊥	⊥
⊥	⊤	⊤
⊥	⊥	⊤

A conditional is true if either its consequent is true or its antecedent is false, and is false otherwise.

The truth-functional analysis of the conditional that we have just adopted is often called the *material conditional;* in our logical symbolism it is represented by the horseshoe "$\supset$". Note that we have so defined "$\supset$" that "$p \supset q$" agrees in its truth-functional behavior with "$-(p.-q)$". That is, to assert a conditional is precisely to deny that the antecedent is true while the consequent is not. In asserting statement (1) above I do no more and no less than commit myself to the falsity of

Today is Tuesday and we are not in Paris.

This consequence of our adoption of the material conditional as an analysis of "if-then" is natural and intuitive, and speaks in favor of that analysis. (The equivalence of "$p \supset q$" and "$-(p.-q)$" also shows that the symbol "$\supset$" is technically superfluous; we could make do with negation and conjunction. We use "$\supset$" solely for convenience.)

Further support for the analysis of conditionals as material conditionals comes from *generalized conditionals.* The statement "Every number divisible by four is even" can be rephrased

(2) No matter what number x may be, if x is divisible by four then x is even.

That is, the statement can be viewed as affirming a bundle of conditionals:

If 0 is divisible by four then 0 is even
If 1 is divisible by four then 1 is even
If 2 is divisible by four then 2 is even

and so on. The interpretation of each of these individual conditionals as material conditionals is just what we need if (2) is to come out true. For among the conditionals in the bundle we find some with true antecedent and true consequent, some with false antecedent and true consequent, and some with false antecedent and false consequent. But we fail to find any with true antecedent and false consequent. Thus each individual conditional, construed as a material conditional, is true. Hence (2) is true, which is exactly what we want. Moreover, each conditional in the bundle amounts to a statement of the form "–(n is divisible by four . n is not even)". This bundle of negated conjunctions can then be summed up by

> No matter what number x is, it is not the case both that x is divisible by four and x is not even

or, more briefly,

> No number is divisible by four yet is not even.

This is a perfectly accurate reformulation of (2).

Generalized conditionals will be treated more fully in Part II below. They are mentioned here only as an illustration of the central role that the material conditional will play in the analysis of more intricate logical forms.

To be sure, we do not claim that the material conditional is accurate to all uses of "if-then". In particular, a conditional whose antecedent is in the subjunctive mood cannot be analyzed as a material conditional. Prominent among subjunctive conditionals are the counterfactual ones, for example,

> If Robert Kennedy had not been assassinated, he would have become President.

This is called a counterfactual conditional because its antecedent is already assumed to be false. Clearly, then, such conditionals do not behave like material conditionals (nor do we take toward them the everyday attitude of ignoring them once the antecedent is seen to be false). Indeed, they are not truth-functional at all—for, obviously, ordinary usage demands that some counterfactual conditionals with false consequents be true and some with false consequents be false.

Thus we intend the material conditional as an analysis only of indicative conditionals. Even here objections are sometimes raised, on the grounds that a material conditional can be true even though the antecedent is completely irrelevant to the consequent. For example, conditionals like "If Manet was a poet then 2 + 2 = 4" and "If Manet was a poet then 2 + 2 = 5" are true. This might well seem bizarre. Yet it would also be bizarre to call these conditionals false. It is, rather, the conditionals themselves that are strange. Conditionals like these simply play no role in practice. No one would think it worthwhile to assert a conditional when the truth-values of its constituents are already known. In practice, we assert "if p then q" if we do not know the individual truth-values of "p" and of "q", but we have some reason for believing that "p.–q" is not the case. Usually this occurs only when we believe "p" and "q" are somehow connected. Without such a connection, we would never have a reason to frame the conditional at all. That is why the above conditionals seem so odd. Such a connection is needed for the usefulness of a conditional; but that is not to say that such a connection has anything to do with the sense of the conditional. "If p then q" can be taken in the sense of the material conditional, regardless of whether "if p then q" is useful, or used at all. It is not for logic to tell us which conditionals are likely to be uttered. And it is essential to logic that any two statements be allowed to join into a conditional.

A last source of hesitancy in adopting the material conditional comes from a mistaken reading of "$p \supset q$" as "p implies q", rather than as "if p then q". Indeed, it is just wrong to claim that "Manet was a poet" implies "2 + 2 = 5". Moreover, the word "implies" is simply incorrect as a reading of "$\supset$" and should therefore be avoided. In §11 we shall examine the correct use of "implies" and see that, although implication is linked to the conditional, "if-then" and "implies" are notions of quite distinct content.

We adopt the material conditional as a rendering of "if ... then" because it is useful. It will become clear, as we proceed, how appropriate this concept is for many purposes which in ordinary English would be served by "if ... then". We have already seen a particularly good example of this: the material conditional is precisely what is wanted as an analysis of the individual instances covered by a generalized conditional.

To repeat, then: the conditional "$p \supset q$" is true in all cases but that in which "p" is true and "q" is false. Thus "$p \supset q$" agrees with "$-(p.-q)$", and also with "$-p \vee q$". Less obvious, perhaps, is that "$p \supset q$" and "$-q \supset -p$" amount to the same, for the latter is false if and only if "$-q$" is true and "$-p$" is false; that is to say, if and only if "q" is false and "p" is true; and this is exactly the one case in which "$p \supset q$" is false. The conditional "$-q \supset -p$" is called the *contrapositive* of "$p \supset q$". A moment's reflection should convince one that the equivalence of a conditional and its contrapositive is intuitively grounded as well. For example, in affirming statement (1) above, I clearly stake myself also to its contrapositive,

If we are not in Paris then today is not Tuesday.

However, the conditional "$p \supset q$" is not at all the same as its *converse*, the conditional "$q \supset p$". Statement (1) may perfectly well be true, while its converse,

If we are in Paris then today is Tuesday,

is not. (Suppose, for example, that we are in Paris, and the day is Wednesday.) Finally, the conditional "$-p \supset -q$" is called the *inverse* of "$p \supset q$". The inverse is the contrapositive of the converse "$q \supset p$", and so is equivalent to the converse, and not to "$p \supset q$".

Other locutions can do the same work as "if-then". Sometimes we might put the antecedent second, as in "q if p", "q provided that p", and "q in case p". "If p then q" is also synonymous with "p only if q". To see why, note that "p only if q" says, essentially, that if "q" fails then so will "p", that is, that "$-q \supset -p$" holds. Since "$-q \supset -p$" is the contrapositive of, and hence agrees with, the conditional "$p \supset q$", the locution "p only if q" can be equated with "if p then q".

What, then, of the expression "p if and only if q", which has already sometimes been used in this text? This amounts to "(p only if q).(p if q)", that is, to "$(p \supset q).(q \supset p)$". We use "$\equiv$" for "if and only if"; that is, "$p \equiv q$" is just the same as "$(p \supset q).(q \supset p)$". We call statements of the form "$p \equiv q$" *biconditionals*. Aside from "p if and only if q", often abbreviated "p iff q", another locution for "$p \equiv q$" is "p when and only when q", and "p just in case q". The truth-table for biconditional looks like this:

p	q	$p \equiv q$
⊤	⊤	⊤
⊤	⊥	⊥
⊥	⊤	⊥
⊥	⊥	⊤

The biconditional of two statements is true if both the statements are true or if both are false, and is false otherwise.

With the biconditional we have come to the last of our basic truth-functional connectives. Indeed, biconditional, like the conditional, can be dispensed with in favor of negation, conjunction, and disjunction: "$p \equiv q$" amounts to "$(p.q) \vee (-p.-q)$". But, as with the conditional, it is far more convenient to have a symbol for this truth-function.

§8. Logical Paraphrase

Pure logic concerns the abstract properties of and relations among compounds formed by means of the logical connectives. But ordinarily the statements to which we wish to apply logical laws are not themselves written in logical notation. To apply logic we must remedy this: we must paraphrase the given statements using our logical symbolism. Paraphrasing reduces varied idioms of everyday language to a regularized notation, and thus enables us to exhibit in a uniform way the relevant structural features of the statements under consideration.

Logical paraphrase requires three basic tasks:

(1) the locutions that serve as connectives have to be identified and suitably translated into symbols;

(2) the constituents of the statement have to be demarcated, and possibly rephrased to make their content explicit;

(3) the organization of the constituents, that is, the grouping, has to be determined.

Throughout this process, we must rely principally on our sense for everyday language. We must arrive at an understanding of what the statement is meant to convey, and then judge whether any suggested paraphrase does justice to the original. There are few hard-and-fast rules, for we rely on a large variety of idiomatic features.

As for the first task, we have already discussed many of the words that can usually be translated as truth-functional connectives. "And" goes into ".", "or" and "either ... or" into "$\vee$", "not" and "it is not the case that" into "$-$", "if ... then" and "only if" into "$\supset$", and "if and only if" into "$\equiv$". We have also discussed some exceptions to these rules.

There are a few locutions not already mentioned that may also serve to express truth-functional connectives. Conjunctions can be expressed not just by "and" but also by "but", by "although", and by punctuation. The differences among "and", "but", and "although" are rhetorical rather than logical. Each of the statements

> Churchill voted "Aye" and Asquith voted "Nay"
> Churchill voted "Aye" but Asquith voted "Nay"
> Churchill voted "Aye" although Asquith voted "Nay"

is true just in case both constituents are true. We would use the statement with "but" if we wish to emphasize the contrast between the divergent votes, and we would use "although" if the contrast is dramatic and surprising—for example, if Churchill had previously always agreed with Asquith. Use of one rather than another expresses an attitude toward the relevant facts, but involves no difference in truth-value.

Turning now to the conditional, we mentioned in the preceding section a number of variant locutions for "if *p* then *q*". To these let us add one more, namely, "not *p* unless *q*". For example, the statement

> The senator will not testify unless he is granted immunity

can be identified with

> The senator will testify only if he is granted immunity,

that is, with

> The senator will testify $\supset$ the senator is granted immunity.

This seems fair enough, but it has a curious consequence. To identify "not *p* unless *q*" with "$p \supset q$" is at the same time to identify it with "$-p \vee q$". Thus we are taking "unless" to amount to "$\vee$", and hence to "or". This may seem odd, but in the end the oddity arises for the same reasons as with the analysis of "if-then" as "$\supset$". That is, "*p* unless *q*" may seem to suggest some connection between "*p*" and "*q*". As before, though, connections between "*p*" and "*q*" might underlie the usefulness of the statement "*p* unless *q*", but need not be taken to enter into the sense of the statement.

Incidentally, it can be argued that "unless"—like "or" itself—is occasionally used in an exclusive sense. "I'll go to the party unless Mother objects" might (depending on context) be justifiably interpreted as affirming both conditionals "If Mother doesn't object then I'll go to the party" and "If

Mother objects then I won't go to the party", rather than simply the former conditional. The conjunction of these two conditionals amounts to "Either I'll go to the party or Mother objects, but not both". Matters here are not as clear as with "or", but in any case we shall interpret all uses of "unless" below inclusively.

So far we have been discussing task (1), the correlation of words of ordinary language with the truth-functional connectives. Task (2) is less straightforward, since we may need not only to translate connectives but also to rephrase the constituent statements. Of course, this necessity arises if the original statement is condensed, for then omitted words have to be reinstated in the paraphrase. Thus, to paraphrase "Fred sang and danced" we must not only replace "and" with "$.$", but also insert the missing "Fred" before "danced". Subtler, however, are those cases where rephrasing is needed to prevent changes of meaning within the statement or group of statements being considered. We have seen a case of this in §2, concerning pronouns. Since the constituents are to be thought of as independent statements, and hence insulated from each other, we cannot let stand in one constituent any pronoun whose interpretation is fixed by a noun in another constituent. Such pronouns must be replaced by their antecedents. For similar reasons, the two conjunctions

> Acheson counseled restraint and Truman agreed
>
> MacArthur argued for invasion and Truman did not agree

cannot in one and the same breath be taken to have the logical forms "$p.q$" and "$r.-q$"; for then it would follow that they cannot both be true. Of course they can both be true.

The point is that the second conjuncts of the two conjunctions cannot be taken as "Truman agreed" and "–(Truman agreed)"; rather, they must be fleshed out to "Truman agreed with Acheson" and "–(Truman agreed with MacArthur)".

Logical analysis requires that the same expression always be given the same interpretation in the course of a single stretch of discourse. Violation of this principle was traditionally known as the *fallacy of equivocation*. As mentioned in §1, we allow ourselves to use sentences that are not, strictly speaking, statements, on the assumption that the context acts uniformly on the statements under consideration at one time. The fallacy of equivocation can occur when the interpretation of a context-dependent expression is not settled by the overarching context, but is influenced in varying ways by immediate contexts. In such cases we have to rephrase, in order to eliminate this influence of immediate context.

The third task we have singled out as crucial to paraphrasing is that of determining the intended grouping. In §6 we discussed some of the ways that grouping can be indicated in ordinary language. The following example illustrates other clues that can help:

(1) If Figaro does not expose the Count and force him to reform, then the Countess will discharge Susanna and resign herself to loneliness.

The words "if" and "then" obviously mark out the antecedent of the conditional. Moreover, the condensation of the clause after "then" shows that this whole clause, not merely the clause "the Countess will discharge Susanna", must be the consequent of the conditional. Thus (1) is to be paraphrased as a conditional with antecedent,

> Figaro does not expose the Count and force him to reform,

and consequent,

> the Countess discharges Susanna . the Countess resigns herself to loneliness.

Now we attack the antecedent. It has, as constituents, the statements "Figaro exposes the Count" and "Figaro forces the Count to reform" (note the necessity of replacing the pronoun "him"). But does it have form "$-p.q$" or the form "$-(p.q)$"? Again, the condensation of the two conjuncts shows that the form is the latter. Thus (1), fully paraphrased, is

> –(Figaro exposes the Count . Figaro forces the Count to reform) ⊃ (the Countess discharges Susanna . the Countess resigns herself to loneliness).

That is, (1) has the logical form "$-(p.q) \supset (r.s)$".

When a statement is complex, the best strategy in paraphrasing is to look for the outermost structure first and then *paraphrase inward* step by step. Each step then yields smaller structures that can be analyzed further. Let us treat the following example:

(2) The trade deficit will diminish and agriculture or telecommunications will lead a recovery provided that both the dollar drops and neither Japan nor the EU raise their tariffs.

First we seek the main connective of (2). Here there is a choice: either (2) is a conjunction, whose main connective is the first "and", or it is a conditional, whose main connective is "provided that". The latter seems more plausible, so we choose it, obtaining a conditional:

(3) (both the dollar drops and neither Japan nor the EU raise their tariffs) ⊃ (the trade deficit diminishes and agriculture or telecommunications leads a recovery).

We now treat the antecedent and the consequent as two separate problems. The main connective of the antecedent is, clearly, "and". Thus the antecedent becomes

> (the dollar drops) . (neither Japan nor the EU raise their tariffs).

Recall that "neither *p* nor *q*" can be rendered "–*p* . –*q*"; the second conjunct can be paraphrased as

> –(Japan raises its tariffs) . –(the EU raises its tariffs).

Now we turn to the consequent of (3). Its main connective is "and"; it is a conjunction:

> (the trade deficit diminishes) . (agriculture or telecommunications leads a recovery).

The second conjunct is, obviously, a disjunction. Thus, we finally obtain

> ((the dollar drops) . –(Japan raises its tariffs) . –(the EU raises its tariffs)) ⊃ ((the trade deficit diminishes) . (agriculture leads a recovery ∨ telecommunications leads a recovery)).

Thus, the truth-functional form of statement (4) is "$(p.-q.-r) \supset (s.(t \vee u))$".

B. Logical Assessment

§9. Schemata and Interpretation

We have been using the letters "p", "q", "r", and so on to represent statements, and have been looking at expressions like "$p \vee q$" and "$(p . q) \supset r$", which represent compound statements. We call "p", "q", ... *sentence letters*, and the compounds constructed from them and the truth-functional connectives *truth-functional schemata*. (In Part I we shall usually omit the modifier "truth-functional", since schemata of other kinds won't be encountered until Part II.) Schemata are not themselves statements. Their constituents, the sentence letters, state nothing, but are mere stand-ins for statements. Schemata are logical diagrams of statements, diagrams obtained by abstracting from all the internal features of the statements save those relevant to the logical structures with which we are concerned.

An *interpretation* of sentence letters is a correlation of a statement with each of the sentence letters. Given such a correlation, a schema constructed from the sentence letters is interpreted by replacing each letter with its correlated statement. Thus, under the interpretation that correlates "Figaro exulted" with "p", "Basilio fretted" with "q", and "the Count had a plan" with "r", the schema "$p.(q \vee r)$" becomes the statement

(1) Figaro exulted . (Basilio fretted ∨ the Count had a plan)

or, in ordinary language,

(2) Figaro exulted and either Basilio fretted or the Count had a plan.

To say that a statement has the logical form given by a certain schema, or that the statement is *schematized* by the schema, is just to say that there is an interpretation under which the schema becomes the statement (or, more pedantically: there is an interpretation under which the schema becomes a paraphrased form of the statement).

Clearly many statements can be schematized by the same schema, since sentence letters may be interpreted in infinitely many ways. Moreover, a single statement may often be schematized by different schemata. Trivially, every statement can be schematized by a sentence letter standing alone, although such a schematization is not very informative. A statement like (2) above can also be schematized by "$p.q$", since "$p.q$" becomes (2) when "p" is interpreted as "Figaro exulted" and "q" is interpreted as "Basilio fretted or the Count had a plan". And, as we have seen, (2) can also be schematized by "$p.(q \vee r)$". The difference here is one of *depth* of analysis; the last schematization is the most informative, since it displays more of the truth-functional structure. It shows not just that (2) is a conjunction, but also that the second conjunct is a disjunction. When in the future we speak of *the* schematization of a statement, we mean the schema that displays all of the evident truth-functional structure of the statement.

There is another sense in which sentence letters may be interpreted: namely, by assigning a truth-value to each. Under any such assignment to its sentence letters, a schema comes out either true or false; which truth-value the schema has may easily be calculated. Interpretations in this sense are called *truth-assignments*. Truth-assignments are more direct than interpretations in the first sense when our concern is to calculate the truth-value behavior of compounds, for since the connectives are truth-functional, the truth-values of the constituents are all that matter to the truth-value of the whole. To say that a schema comes out true, for example, under the truth-assignment that assigns truth to "p" and to "q" and falsity to "r" is just to say that the schema comes out true under any interpretation in the first sense that correlates true statements with "p" and with "q" and a false statement with "r".

Truth-tables may be used to display what the truth-value of a schema is under each truth-assignment to its sentence letters. The truth-table for "$(p \vee q) \supset r$" is

p	q	r	$(p \vee q) \supset r$
⊤	⊤	⊤	⊤
⊤	⊤	⊥	⊥
⊤	⊥	⊤	⊤
⊤	⊥	⊥	⊥
⊥	⊤	⊤	⊤
⊥	⊤	⊥	⊥
⊥	⊥	⊤	⊤
⊥	⊥	⊥	⊤

In it, each of the eight lines represents one of the eight truth-assignments to "p", "q", and "r". On each line the entry in the final column shows what the truth-value of "$(p \vee q) \supset r$" is under that truth-assignment.

In general, there are 2^n truth-assignments to n sentence letters; hence the truth-table for a schema constructed from n sentence letters will contain 2^n lines. The order of lines is immaterial, so long as each truth-assignment is represented. But for practical reasons—familiarity and ease of comparison—we always arrange the lines in a canonical order. The order, for sentence letters "p", "q", and "r", may be described thus: the first four lines have "p" true, and the second four have "p" false; in each of these halves, four lines apiece, the first two lines have "q" true and the second two have "q" false; in each of these quarters, two lines apiece, the first line has "r" true and the second "r" false. In other words: in the column headed by "r", "$\top$" and "$\bot$" alternate on every line; in the column headed by "q", they switch every second line; in the column headed by "p", they switch every fourth line.

Similarly, if there are four sentence letters "p", "q", "r", and "s", there will be sixteen lines, since $16 = 2^4$. In the canonical order, the first eight lines have "p" true and the second eight have "p" false; the truth-value of "q" switches every four lines, with "q" true on the first four; the truth-value of "r" switches every two lines, with "r" true on the first two; the truth-value of "s" switches at each line, with "s" true on the first line.

To obtain the entries in the final column of the truth-table requires calculation. For example, on the fourth line of the truth-table for "$(p \vee q) \supset r$", a line that represents the assignment of truth to "p" and falsity to "q" and to "r", we note first that "$p \vee q$" is true under this assignment, since at least one of its disjuncts is true. Then we note that since "r" is false and a conditional with true antecedent and false consequent is

false, the schema "$(p \vee q) \supset r$" is false. Hence we enter "$\bot$" on this line in the final column.

This sort of calculation must be done for each line: first we calculate the truth-value of "$p \vee q$", and then we use that value in a calculation of the truth-value of "$(p \vee q) \supset r$". To keep track, it is convenient to expand the truth-table by interposing a column headed by "$p \vee q$". Our first step would then be to fill in this column completely. This is easily done by inspection: "$\top$" is entered on all lines but those that contain "$\bot$" both under "p" and under "q". The partially completed truth–table looks like this:

p	q	r	$p \vee q$	$(p \vee q) \supset r$
$\top$	$\top$	$\top$	$\top$	
$\top$	$\top$	$\bot$	$\top$	
$\top$	$\bot$	$\top$	$\top$	
$\top$	$\bot$	$\bot$	$\top$	
$\bot$	$\top$	$\top$	$\top$	
$\bot$	$\top$	$\bot$	$\top$	
$\bot$	$\bot$	$\top$	$\bot$	
$\bot$	$\bot$	$\bot$	$\bot$	

We may now easily fill in the final column: we enter "$\bot$" on all lines that contain "$\top$" under "$p \vee q$" and "$\bot$" under "r", and enter "$\top$" on the rest. So we would enter "$\top$" on just the first, third, fifth, seventh, and eighth lines.

Suppose now we wish to construct the truth-table for "$-(p.q) \vee (p \equiv (q.r))$". Again there are three sentence letters, so the truth-table will have eight lines. The first three columns of the truth-table are headed by the sentence letters, and the final column by the whole schema. Intermediate columns are headed by those schemata whose truth-value we must calculate in the course of calculating the truth-value of the whole. Thus the column headings look like this:

p	q	r	$p.q$	$-(p.q)$	$q.r$	$p \equiv (q.r)$	$-(p.q) \vee (p \equiv (q.r))$

We then fill in the columns. The first three are filled in so as to represent the eight truth-assignments in canonical order. The subsequent columns are filled in, one by one, in accordance with the rules governing the truth-functional connectives, namely:

(1) a negation is true if what is negated is false, and is false otherwise;

(2) a conjunction is true if its conjuncts are all true, and is false otherwise;

(3) a disjunction is true if at least one of its disjuncts is true, and is false otherwise;

(4) a conditional is true except if its consequent is false and its antecedent is true; in that case it is false;

(5) a biconditional is true if its two constituents have the same truth-value, and is false otherwise.

Our completed truth-table looks like this:

p	q	r	$p.q$	$-(p.q)$	$q.r$	$p \equiv (q.r)$	$-(p.q) \vee (p \equiv (q.r))$
T	T	T	T	⊥	T	T	T
T	T	⊥	T	⊥	⊥	⊥	⊥
T	⊥	T	⊥	T	⊥	⊥	T
T	⊥	⊥	⊥	T	⊥	⊥	T
⊥	T	T	⊥	T	T	⊥	T
⊥	T	⊥	⊥	T	⊥	T	T
⊥	⊥	T	⊥	T	⊥	T	T
⊥	⊥	⊥	⊥	T	⊥	T	T

It should be clear that our procedure can be applied, in a purely mechanical fashion, to any truth-functional schema whatever.

To be sure, this procedure is somewhat long-winded. On occasion, it may be shortened. For example, it should be clear by inspection that "$-(p.q)$" is true under just those interpretations that assign falsity to one or both of "p" and "q". Hence the column headed by "$p.q$" might be omitted from the truth-table, since the calculation of the entries in the column headed by "$-(p.q)$" may be done directly. In general, the column headed by a schema may be omitted if one does not need to calculate the truth-values of that schema explicitly. This, of course, is a matter of taste: calculations that some prefer to do directly, in their heads, others prefer to carry out in a more explicit, step-by-step manner.

Another shortcut may be also be used. The final column is to give truth-values for a disjunction whose first disjunct is "$-(p.q)$". That disjunction will be true whenever "$-(p.q)$" is true. Thus, after calculating the column headed by "$-(p.q)$" and discovering "T" on the last six lines, we may immediately enter "T" on the last six lines of the final column. We are thus spared the work of calculating the truth-value of "$p \equiv (q.r)$" on those lines; we need calculate this value only on the first two. In the construction of truth-tables there are often opportunities like these for skipping calculations. Again, how much to avail oneself of these opportunities is a matter of personal taste.

Another device, although it saves no steps, can save space. Instead of heading one column with "$q.r$" and another with "$p \equiv (q.r)$", we can use the label "$p \equiv (q.r)$" once, writing the entries for "$q.r$" under the "." and the entries for "$p \equiv (q.r)$" under the "≡". Indeed, we could use the label "$-(p.q) \vee (p \equiv (q.r))$" just once, writing the entries for "$-(p.q)$" under the first disjunct, the entries for "$q.r$" under the last ".", the entries for "$p \equiv (q{.}r)$" under the "$\equiv$", and the entries for the whole schema under the "$\vee$". Of course this means we do not fill in the columns from left to right. Here is the truth-table written in this compressed manner:

p	q	r	$-(p{.}q)$	$\vee$	$(p$	$\equiv$	$(q{.}r))$
$\top$	$\top$	$\top$	$\bot$	$\top$		$\top$	$\top$
$\top$	$\top$	$\bot$	$\bot$	$\bot$		$\bot$	$\bot$
$\top$	$\bot$	$\top$	$\top$	$\top$		$\bot$	$\bot$
$\top$	$\bot$	$\bot$	$\top$	$\top$		$\bot$	$\bot$
$\bot$	$\top$	$\top$	$\top$	$\top$		$\bot$	$\top$
$\bot$	$\top$	$\bot$	$\top$	$\top$		$\top$	$\bot$
$\bot$	$\bot$	$\top$	$\top$	$\top$		$\top$	$\bot$
$\bot$	$\bot$	$\bot$	$\top$	$\top$		$\top$	$\bot$
			1	4		3	2

(The numerals at the bottom indicate the order in which the columns are computed.) It is of course the column under "$\vee$" that we are interested in, for it gives the truth-values for the whole schema. We shall persist in calling the column that gives the truth-values for the whole schema "the final column", even though it may no longer be the rightmost column. Note that if the shortcut suggested two paragraphs ago is used, then there need be no entries in columns 2 and 3 except on the first two rows.

In constructing truth-tables for complex schemata, it is best not to overuse this space-saving device. Too great an attempt at compression harms visual perspicuity, and can lead to confusion of columns.

While on the topic of brevity and perspicuity, let us raise an issue with regard to schemata themselves. In our symbolism, parentheses are used to indicate grouping. Indeed,

as a grouping device they are simple, straightforward, and rigorous. Nevertheless, long schemata containing many parentheses are hard to take in at a glance; to ascertain the structure, we may well have to start counting the parentheses. Hence it is advantageous to adopt conventions that permit some parentheses to be omitted.

One such convention has already been tacitly in use. The negation sign is understood to govern as little as possible of what follows it. Thus "$-p{.}r$" is "$(-p){.}r$", not "$-(p{.}r)$"; similarly, "$-(p \vee q){.}r$" is "$(-(p \vee q)){.}r$", not "$-((p \vee q){.}r)$".

We now adopt another convention: unless other parentheses rule to the contrary, the connective "$\vee$" is to be understood as marking a greater break than "$.$", and the connectives "$\supset$" and "$\equiv$" as marking greater breaks than "$\vee$" and "$.$". Thus we may write

$p{.}q \vee r$	instead of:	$(p{.}q) \vee r$
$p{.}q \vee (r \supset s){.}-(q{.}r)$	instead of:	$(p{.}q) \vee ((r \supset s){.}-(q{.}r))$
$p{.}q \vee r \supset s{.}(p \vee r)$	instead of:	$((p{.}q) \vee r) \supset (s{.}(p \vee r))$
$p{.}q \vee r \equiv s{.}p \vee r$	instead of:	$((p{.}q) \vee r) \equiv ((s{.}p) \vee r)$.

This convention should be used with care. Sometimes applying it too thoroughly results in a decrease rather than an increase of readability. In such cases it is wiser to retain some of the parentheses, even though the convention would allow them to be dropped.

§10. Validity and Satisfiability

A truth-functional schema that comes out true under all interpretations of its sentence letters is said to be *valid*. A truth-functional schema that comes out true under at least one interpretation is said to be *satisfiable*, and one that comes out true under no interpretation is said to be *unsatisfiable*. Thus "$p \supset p$" is valid, "$p . -q$" is satisfiable but not valid, and "$p . -p$" is unsatisfiable. Note that every valid schema is satisfiable: for if a schema comes out true under all interpretations then surely it comes out true under at least one.

To determine whether a schema is satisfiable and whether it is valid, we need only inspect the final column of the truth-table for the schema. The schema is valid just in case each entry in this column is "$\top$"; the schema is satisfiable if at least one entry is "$\top$"; and the schema is unsatisfiable if no entry is "$\top$".

In testing schemata for validity and satisfiability, we need not always construct the whole truth-table. A test for satisfiability can be terminated with an affirmative answer as soon as we find a line of the truth-table that has "$\top$" entered in the final column, and a test for validity can be terminated with a negative answer as soon as we find a line that has "$\bot$" entered in the final column. Thus the schema "$p . q \vee -p . -r \supset (q \equiv r)$" is shown satisfiable but not valid by just the first two lines of its truth-table.

p	q	r	$p . q$	$\vee$	$-p . -r$	$\supset$	$(q \equiv r)$
$\top$	$\top$	$\top$	$\top$	$\top$	$\bot$	$\top$	$\top$
$\top$	$\top$	$\bot$	$\top$	$\top$	$\bot$	$\bot$	$\bot$

However, a positive answer to a test for validity and a negative answer to a test for satisfiability can be obtained only once the entire truth-table is constructed, for validity requires that *all* entries in the final column be "$\top$" and unsatisfiability that *all* be "$\bot$".

We may also speak of valid statements, as well as schemata. A statement is valid if it can be schematized by a valid schema. Actually, to be more explicit we call such statements *truth-functionally valid*, to mark the fact that the schematization at issue is truth-functional (rather than the more intricate kinds we shall investigate later on). Truth-functionally valid statements are in a sense trivially true, for they give us no information about the subject matter of which their constituent statements speak. The truth-functionally valid statement

> If IBM shares are going to rise and Microsoft shares are going to fall, then IBM shares are going to rise

tells us nothing about the stock market; any other statements could replace "IBM shares are going to rise" and "Microsoft shares are going to fall", and the result would still be true. A truth-functionally valid statement is a logical truth: it is true purely by dint of its truth-functional structure, insofar as every statement that shares that structure is likewise true.

§11. Implication

An important task for logic is that of showing whether a statement logically follows from another statement. The statement "All whales are warm-blooded" follows logically from the statement "All whales are mammals and all mammals are warm-blooded"; the statement "Cassius is not both lean and hungry" follows logically from the statement "Cassius is not lean"; and the statement "If Susanna relents then the Count will be happy" follows logically from the statement "If either Susanna relents or Marcellina wins her case then the Count will be happy". The first of these three examples lies beyond the scope of truth-functional logic, but the second and third can be treated with the tools we have developed so far.

That "Cassius is not both lean and hungry" follows logically from "Cassius is not lean" is a matter of the truth-functional form of the two statements. The two statements may be schematized as "$-(p.q)$" and "$-p$", respectively; that is, there is an interpretation in the first sense under which "$-(p.q)$" becomes "Cassius is not both lean and hungry" and "$-p$" becomes "Cassius is not lean". These two schemata have the following relation: there is *no* interpretation under which "$-p$" comes out true and "$-(p.q)$" comes out false.

Similarly, "If Susanna relents or Marcellina wins her case then the Count will be happy" and "If Susanna relents then the Count will be happy" can be schematized as "$p \vee q \supset r$" and "$p \supset r$", respectively; and no interpretation makes the schema "$p \vee q \supset r$" true yet makes the schema "$p \supset r$" false. We phrase the crucial relation between these schemata thus: "$-p$" implies "$-(p.q)$", and "$p \vee q \supset r$" implies "$p \supset r$".

One truth-functional schema implies another if and only if there is no interpretation of the sentence letters under which the first schema is true and the second false. In other words, one schema implies another if and only if every interpretation of the sentence letters they contain that makes the first schema true also makes the second schema true.

Whether a schema X implies a schema Y can be determined by a simple procedure: first construct a truth-table for the two schemata; then see whether there is a line that contains "$\top$" in the column headed by X and contains "$\bot$" in the column headed by Y. If there is no such line, then X implies Y; if there is such a line, then X does not imply Y.

That "$-p$" implies "$-(p.q)$" can thus be seen from the following truth-table, since no line contains both "$\top$" under "$-p$" and "$\bot$" under "$-(p.q)$":

p	q	$-p$	$-(p.q)$
$\top$	$\top$	$\bot$	$\bot$
$\top$	$\bot$	$\bot$	$\top$
$\bot$	$\top$	$\top$	$\top$
$\bot$	$\bot$	$\top$	$\top$

That the schema "$p \vee q$" does not imply the schema "$p.q$" can be seen from the following partial truth-table:

p	q	$p \vee q$	$p.q$
$\top$	$\top$	$\top$	$\top$
$\top$	$\bot$	$\top$	$\bot$

Here we may cease computing, having obtained a line with "$\top$" under "$p \vee q$" and "$\bot$" under "$p.q$". We see that implication does not hold.

To test whether a schema X implies a schema Y is just to test whether the conditional whose antecedent is X and whose consequent is Y is valid. After all, a conditional is valid if and only if no interpretation makes its antecedent true yet makes its consequent false. Thus, implication is validity of the conditional.

So we may show that "$-p$" implies "$-(p.q)$" by showing that every entry in the final column of the truth-table for "$-p \supset -(p.q)$" is "$\top$". Of course, if we are checking for implication, it is just extra work to compute the entries in this final column: inspection of the two columns headed by "$-p$" and "$-(p.q)$" already suffices to settle the mattter. However, it is, on occasion, of practical value to think of implication as validity of the conditional. For example, to see that "$-p$" implies "$-(p.q)$"—that is, that "$-p \supset -(p.q)$" is valid—we might observe that this conditional is the contrapositive of the conditional "$p.q \supset p$". The validity of the latter conditional is obvious. We may conclude that the former conditional is valid. Thus by recognizing certain obviously valid conditionals, we may quickly recognize various cases of implication, without having to construct a truth-table.

In using truth-tables to check for implication, the shortcut mentioned in §9 can be of great use. Suppose we are testing whether a schema X implies a schema Y. Once we enter an "$\perp$" on a line in the column headed by X, we may subsequently ignore that line—we need not compute the truth-value of Y on that line. Similarly, if we choose to compute the truth-values of Y first, once we enter a "$\top$" on a line in the column headed by Y, we may subsequently ignore that line. We are interested solely in seeing whether or not some line contains "$\top$" under X and "$\perp$" under Y.

To check whether "$p \vee q \supset r$" implies "$p \supset r$" we might compute the truth-values of "$p \supset r$" first, obtaining

p	q	r	$p \vee q \supset r$	$p \supset r$
$\top$	$\top$	$\top$		$\top$
$\top$	$\top$	$\perp$		$\perp$
$\top$	$\perp$	$\top$		$\top$
$\top$	$\perp$	$\perp$		$\perp$
$\perp$	$\top$	$\top$		$\top$
$\perp$	$\top$	$\perp$		$\top$
$\perp$	$\perp$	$\top$		$\top$
$\perp$	$\perp$	$\perp$		$\top$

Then we need compute the truth-value of "$p \vee q \supset r$" on just those two lines, the second and the fourth, where "$p \supset r$" is false. Since we come out with "$\perp$" both times, we are assured that the implication holds.

A little thought can serve to shorten matters further. It is apparent by inspection that "$p \supset r$" is true whenever "p" is false. After noting this, we need not bother even to write down the last four lines, those on which "p" is false.

Another example: to check whether "$p.-q$" implies "$(p \supset q) \supset r$", it pays to note first that "$p.-q$" is true just when "p" is true and "q" false. We need therefore consider only two lines:

p	q	r	$p.-q$	$p \supset q$	$(p \supset q) \supset r$
$\top$	$\perp$	$\top$	$\top$	$\perp$	$\top$
$\top$	$\perp$	$\perp$	$\top$	$\perp$	$\top$

Since "$\top$" is entered under "$(p \supset q) \supset r$" on both these lines, the implication holds.

In short, in testing whether X implies Y, we need not write down the lines of the truth-table on which it is evident by inspection that Y is true; nor need we write down those lines on which it is evident that X is false. As always, how far this labor-saving device is to be carried—that is, what should count as being "evident by inspection"—is a matter of individual preference. A test for implication may always be executed purely mechanically, with the full truth-table. But it is usually quicker and more interesting, and indeed it is helpful for a better understanding of truth-functional logic, to attempt to apply some shortcuts.

It is also helpful to develop an ability at recognizing quickly the implications that hold between various simple schemata. For example, with a little practice it should become obvious that "p" implies "$p \vee q$", "$q \supset p$", and "$-p \supset q$"; that "$p.q$" implies "p" and "q"; that "$p \supset q$" implies "$-q \supset -p$", "$p \supset q \vee r$", and "$p.r \supset q$"; and that "$p \supset q$" is implied by "$p \supset q.r$" and by "$p \vee r \supset q$".

As with validity, we may also apply the notion of implication to statements. If one schema implies another, and a pair of statements can be schematized by those schemata, then we may say that the one statement *truth-functionally implies* the second. Thus "Cassius is not lean" truth-functionally implies "Cassius is not both lean and hungry", and "If either Susanna relents or Marcellina wins her case then the Count will be happy" truth-functionally implies "If Susanna relents then the Count will be happy". Truth-functional implication is the relation that one statement bears to another when the second follows from the first by logical considerations within the scope of truth-functional logic, that is, when the second may be inferred from the first by dint of the truth-functional structure of the two statements.

Implication is thus of concern in inference, that is, in logical argumentation. The conclusion of an argument with one premise logically follows from the premise if the premise implies the conclusion, for if it does then we are assured, on logical grounds alone, that if the premise is true the conclusion will also be true. Similarly, the conclusion of an argument from several premises logically follows from the premises if the premises *jointly* imply the conclusion, that is, if the conjunction of the premises implies the conclusion. Hence, to assess an argument one first schematizes the premises and the conclusion, and then checks whether every interpretation that makes all the premise-schemata true also makes the conclusion-schema true. If so, the premises (jointly) imply the conclusion, so that the conclusion does follow from the premises.

Some logic textbooks call an argument "valid" iff its premises imply its conclusion. We do not use this terminology here, in order to avoid confusion with the notion of validity as applied to schemata or to statements. (Such textbooks also call an argument "sound" iff its premises imply its conclusion and its premises are true. Soundness of arguments, in this sense, is no concern of logic, since the actual truth-value of premises is not a logical matter. We use the word "sound" for a different, and logically very important, notion, discussed in §§17 and 35.)

§12. Use and Mention

Implication, as we have seen, is closely related to the conditional: implication holds when and only when the relevant conditional is valid. Unfortunately, this relation between the two notions has too often been taken to license the reading of the sign "$\supset$" as "implies", rather than as "if ... then". That is an error, for it confuses the assertion of a conditional—the assertion, for example, that if Cassius is not lean then Cassius is not both lean and hungry—with the assertion that the conditional has a certain logical property, that is, that the conditional "if Cassius is not lean then Cassius is not both lean and hungry" is true by dint of its truth-functional form. To assert that a statement implies another statement is to do more than affirm a conditional; it is to state that the conditional is logically true—true by dint of the logical structure of the two statements.

To assert that a statement implies another is thus to state something about those statements. Implication is a relation between statements (or between schemata). On the other hand, to assert a conditional is not to state anything about the constituent statements of the conditional. If I assert

If the senator is granted immunity then he will testify,

then I am talking not about statements, but about the senator. "If-then" is not a relation between statements, any more than is "and".

To make this clearer, we must reflect on the distinction between *use* and *mention.* We use words to talk about things. In the statement

(1) Frege devised modern symbolic logic,

the first word is used to refer to a German logician. The statement *mentions* this logician and *uses* a name to do so. Similarly, in the statement

(2) The author of *Foundations of Arithmetic* devised modern symbolic logic,

the sentence mentions the same logician, and uses the expression consisting of the first six words to do so. The first six words of sentence (2) constitute a *complex name,* a name of Gottlob Frege. In general, to speak of an object we use an expression that is a name of that object, or, in other words, an expression that refers to that object. Clearly the object mentioned is not part of the statement: its name is.

Confusion can arise when we speak about linguistic entities. If I wish to mention—that is, to talk about or refer to—an expression, I cannot use that expression; for if I did I would be mentioning the object that the expression refers to. Instead, I must use a name of that expression. Thus I might say:

(3) The first word of statement (1) is a name of a German logician.

The first six words of statement (3) constitute a name of an expression. If we wish to obtain a truth from "______ is a name of a German logician", we must fill in the blank not with a name of a German logician but with the name of a name of a German logician.

Logicians adopt a simple convention for constructing names of expressions: a name for an expression is formed by surrounding the expression with quotation marks. (We use double quotation marks; other authors use single ones. Also, if the expression to be named is displayed on an isolated line or lines, we let the isolation do the work of the quotation marks.) Thus,

"Frege" is a name of a German logician
"Frege" refers to Frege.

That is, in statement (1) we use the first word to mention Frege; the first word is "Frege"; in statement (2) we use the first six words to mention Frege; these words are "the author of *Foundations of Arithmetic*".

Similarly, when we wish to mention (talk about) a statement, we use a name of the statement. We might say

Statement (1) is true

or

"Frege devised modern symbolic logic" is true.

A schema is an expression; hence to talk about (mention) a schema we must use a name of the schema:

"$p \vee q$" is a schema
"$p \vee -p$" is valid

"p" implies "$p \vee q$"

"p" implies the disjunction of "p" and "q"

Note that the last six words of the last example constitute a complex name of the schema "$p \vee q$".

In sum, to obtain a sentence that says that one statement implies another, or that one schema implies another, we must use the word "implies" between *names* of the two statements or schemata. The resulting sentence uses those names to mention the statements or schemata.

On the other hand, when we compound a statement or schema from two others by means of "if-then", or "$\supset$", we use the statements or schemata themselves and not their names. We do not *mention* the statements or schemata; there is no reference to them; they merely occur as parts of a longer statement or schema.

Thus in talking about a schema we use a name of that schema, most usually the name obtained by surrounding the schema with quotation marks. What then do we use to talk about schemata generally? Naturally, we use expressions such as "every schema" and the like; but for some purposes we must also use variables that range over schemata, which are called *syntactic variables*. Just as we might use "x" as a numerical variable and say

(4) A number x is odd if and only if x^2 is odd,

we can use "X" and "Y" as syntactic variables and say

(5) A schema X implies a schema Y if and only if the conditional with antecedent X and consequent Y is valid.

Particular instances of (4) are obtained by replacing the variable "x" with a name of a number, for example, "3 is odd if an only if 3^2 is odd". So too particular instances of (5) are obtained by replacing the variables "X" and "Y" with names of schemata, for example,

(6) "$-p$" implies "$-(p.q)$" if and only if the conditional with antecedent "$-p$" and consequent "$-(p.q)$" is valid.

Note that the last ten words of (6) constitute a complex name of the schema "$-p \supset -(p.q)$".

§13. Equivalence

Two truth-functional schemata are equivalent if they have the same truth-value under every interpretation of their sentence letters. We have already pointed out many simple cases of equivalence:

"p" to "$--p$"

"$p.q$" to "$q.p$" and "$p \vee q$" to "$q \vee p$"

"$-(p.q)$" to "$-p \vee -q$" and "$-(p \vee q)$" to "$-p.-q$"

"$p \supset q$" to "$-(p.-q)$", to "$-p \vee q$", and to "$-q \supset -p$"

To test two schemata for equivalence, we need only construct a truth-table for the schemata and see whether on each line the same value is entered in the two columns headed by the schemata. Thus, by comparing the appropriate columns of the following truth-table, we see that "$p \supset (q \supset r)$" is equivalent to "$p.q \supset r$":

p	q	r	$p \supset$	$(q \supset r)$	$p.q$	$\supset r$
T	T	T	T	T		T
T	T	⊥	⊥	⊥	T	⊥
T	⊥	T	T	T		T
T	⊥	⊥	T	T	⊥	T
⊥	T	T	T			T
⊥	T	⊥	T		⊥	T
⊥	⊥	T	T			T
⊥	⊥	⊥	T		⊥	T

(We have used some shortcuts here. We have not bothered to compute the truth-value of "$q \supset r$" on those four lines where "p" is false; for when "p" is false the conditional "$p \supset (q \supset r)$" is true. Similarly, on those lines where "r" is true, we have immediately entered "T" under "$p.q \supset r$", without bothering to compute the truth-value of "$p.q$".)

Similarly, to see that "$p \supset q$" is not equivalent to "$q \supset p$" we need only inspect the partial truth-table:

p	q	$p \supset q$	$q \supset p$
T	T	T	T
T	⊥	⊥	T

Here we stop, having obtained a disagreement between the truth-value of "$p \supset q$" and that of "$q \supset p$".

It should be clear that two schemata are equivalent if and only if the biconditional of the two schemata is valid, for the biconditional between X and Y is valid just in case every truth-assignment gives the same truth-value to X as it does to Y. Equivalence is the validity of the biconditional. It should be equally clear that two schemata are equivalent just in case they imply each other. *Equivalence is mutual implication.*

Equivalence can be used to license the transformation of one statement into another statement that "says the same thing". Since the schema "$p \supset (q \supset r)$" is equivalent to the schema "$p.q \supset r$", we are fully justified in transforming

(1) If Marcellina loses the case, then Figaro will exult provided that Susanna remains faithful

into

(2) If Marcellina loses the case and Susanna remains faithful, then Figaro will exult

or vice versa. Statements that, like (1) and (2), can be schematized by equivalent truth-functional schemata can themselves be said to be truth-functionally equivalent. Truth-functionally equivalent statements "say the same thing", purely by dint of their truth-functional form.

PART II

MONADIC QUANTIFICATION THEORY

A. ANALYSIS

§18. Monadic Predicates and Open Sentences

Truth-functional logic analyzes statements insofar as they are compounded from simpler statements, and charts out the behavior of such compounds in terms of the behavior of their constituents. Truth-functional logic, however, can yield no account of arguments like these:

> All philosophers are wise.
> Frege is a philosopher.
> Therefore, Frege is wise.

> All philosophers are wise.
> Some philosophers are logicians.
> Therefore, some logicians are wise.

For the premises and conclusions of these arguments are all truth-functionally simple: none of them is a compound of simpler statements. Yet, intuitively, in each case the conclusion does follow logically from the premises. To handle these arguments, analysis must be pressed further. We must examine the construction of statements from components that are not themselves statements. Indeed, the cogency of these arguments rests on such subsentential components, in particular, on the use of the words "some" and "all", and on the multiple occurrence of words like "wise" and "philosopher". Let us consider the latter first.

The statements "Frege is wise", "Socrates is wise", and "The Queen of England is wise" clearly have something in common. They share the words "is wise" as well as the following structural feature: each statement is obtained by putting a name of a particular object in front of "is wise". Thus, we may write what is in common as "______ is wise", where the blank shows where the name of a particular object is to go. Each of these statements serves to ascribe wisdom to a particular object. We can view the task of specifying the object as done by the name, and the task of ascribing wisdom as done by the common part "______ is wise".

This notation, however, is somewhat impractical. Blanks are easily overlooked; moreover, later on we shall need different sorts of blanks. Instead, we use a *placeholder:* a sign that marks an empty place into which names can be put. For now we use as a placeholder the sign "①". Thus we write what is common to our three statements as "① is wise". This expression is called a *monadic predicate.* In general, a monadic predicate is an expression that contains the placeholder "①" and that becomes a statement when the placeholder is supplanted by a name of an object. Monadic predicates are artificial expressions; they do not occur in sentences as is, but only when empty places indicated by the placeholder have been filled in. As Frege put it, monadic predicates are "incomplete".

Not being statements, monadic predicates are neither true nor false. Rather, a monadic predicate is true and false *of* particular objects. "① is wise" is true of all wise individuals and is false of everything else. In particular, then, it is true of King Solomon and false (most would agree) of King George III. It should be clear that a monadic predicate is true of an object if and only if a true statement is obtained by putting a name of the object for the placeholder in the monadic predicate.

It is easy to think of other monadic predicates. Of these

① is a logician

① revolves around the earth

① is an even positive integer,

the first is, for instance, true of Frege and the author of this text, but is false of Maria Callas; the second is true of the moon but false of the sun; and the third is true of 10 and of 1270, but false of 17 and of the Eiffel Tower. In all our examples so far, the placeholder has occupied the grammatical subject place, but this is not necessary. "①'s birthday is in January" is a perfectly good monadic predicate, true of Martin Luther King, Jr. and false of George Washington. "The Eiffel Tower is taller than ①" is true of all people and of the White House, but is false of the Empire State Building. Nor need a placeholder occur only once in the predicate:

① respects ①

Everyone who knows ① likes ①

① is a logician . ① is German

are all monadic predicates. The first of these is true of all self-respecting individuals, and of no others. The last of these is, for instance, true of Frege and false of the author of this text, of Ludwig van Beethoven, and of Maria Callas. Of course, when the placeholder in a monadic predicate is replaced by a name, it must be so replaced in all its occurrences.

Monadic predicates, to repeat, are true and false of objects. Some are true of all objects, like "① is self-identical" (that is, "① = ①") and like "–(① is red . –(① is red))"; some are true of none, like "① is a natural satellite of Venus" and "① is an even prime number greater than 2"; and the rest are somewhere in between. The *extension* of a monadic predicate is the class of objects of which it is true. Thus, the extension of "① is a North American city more populous than Chicago" is the class whose members are Los Angeles, Mexico City, and New York; that of "① is an even positive integer" is the class whose members are 2, 4, 6, 8, and so on. The extension of "① is a natural satellite of the earth" is the class whose one and only member is the moon; and the extension of "① is a natural satellite of Venus" is the class with no members, that is, the empty class. Different predicates can have the same extension: witness "① is a natural satellite of Venus" and "① is an even prime number greater than 2"; or "① is an animal with a heart" and "① is an animal with kidneys". Predicates that possess the same extension are said to be *coextensive.* Otherwise put, two monadic predicates are coextensive if and only if they are true of just the same objects. The sentences we shall construct from monadic predicates are all *extensional:* their truth-values depend only on the extensions of the monadic predicates. That is, in such a sentence a monadic predicate may be replaced by any coextensive one without affecting the truth-value of the whole.

As we have seen, statements can be constructed by putting names for the placeholder in monadic predicates. However, in the arguments we are now studying, particular names are of no importance. Consider, for example,

All philosophers are wise.
Frege is a philosopher.
Therefore, Frege is wise.

That the name "Frege", rather than any other, occurs here plays no role in the cogency of the argument. From a logical point of view, we might as well write the argument thus:

All philosophers are wise.
x is a philosopher.
Therefore, x is wise.

Here "x" is a *variable.* Variables are used, in a sense, as arbitrary names.

Now the sentence "x is wise" is the result of putting the variable "x" for the placeholder in "① is wise". This sentence is not a statement; it is called an *open sentence.* We may think of an open sentence as leaving open what "x" is to stand for. Until a value is assigned to the variable, "x is wise" is neither true nor false; but once such an assignment is made, the truth-value is determined. The closest analogue in ordinary English to an open sentence is a sentence containing a pronoun without an antecedent, like "She is wise". "She is wise" has no truth-value until the person to whom "she" refers is in some way given. Ordinarily, of course, this is given by the context in which "She is wise" occurs. As we shall see, the way we treat variables in open sentences also depends on context. Like pronouns too, variables in open sentences are used for cross-referencing. In the argument above, the use of the same variable "x" in the second premise and the conclusion indicates that the same value must be assigned to the variable in these two sentences.

To sum up, then, open sentences are expressions like statements but for containing variables instead of names. Open sentences, however, are not themselves statements. Rather, they are true or false for particular values of their variables. For example, "x is wise" is true when "x" has value King Solomon, and is false when "x" has value King George III. We sometimes put this as follows: the assignment of King George III to "x" (as its value) makes "x is wise" false. Clearly, if an open sentence is obtained from a monadic predicate by putting "x" for the placeholder, then an assignment to "x" makes the open sentence true just in case the monadic predicate is true of the individual assigned as value to "x".

The utility of open sentences rests on the fact that they behave like statements once an assignment of values to the variable is fixed. In particular, complex open sentences can be constructed by means of the truth-functional connectives. "x is a philosopher . –(x is wise)" is made true by an assignment to "x" just in case that assignment makes "x is a philosopher" true and makes "x is wise" false. Similarly, "x is a philosopher ⊃ x is wise" is made true by an assignment just in case "x is a philosopher" is made false or "x is wise" is made true by that assignment, that is, just in case either the individual assigned as value to "x" is not a philosopher or else that individual is wise.

Note that variables other than "x" may occur in open sentences. (We shall also use "y", "z", "w", "x'", and so on.) The open sentence "x is a philosopher . –(y is wise)" has a truth-value only once values are assigned to both "x" and "y". These values may be the same or different, but, in any case, the open sentence is true if and only if the value assigned to "x" makes "x is a philosopher" true and the value assigned to "y" makes "y is wise" false. Thus, which variable occurs in an open sentence can make a difference. Although "x is wise" and "y is wise" behave similarly when each is taken by itself, "x is a philosopher . –(x is wise)" and "x is a philosopher . –(y is wise)" are quite different. If in fact all philosophers are wise, then no assignment makes the former open sentence true. But since different values may be assigned to "x" and "y", some assignment does make the latter open sentence true. This is another aspect of the use of variables for cross-referencing. Further examples of this phenomenon will occupy us in later sections. For now, however, we shall be concerned principally with one-variable sentences.

§19. The Existential Quantifier

A close relation can easily be discerned between the statement

(1) There is a building that is over 1200 feet tall

and the open sentence

(2) x is a building . x is over 1200 feet tall.

Namely, (1) is true if and only if some value for "x" makes (2) true. We highlight this relation by using the *existential quantifier* "(∃x)", which is read "there is an x such that" or "there exists an x such that". Thus statement (1) can be paraphrased

(3) (∃x)(x is a building . x is over 1200 feet tall).

Note that in saying "there is an object x such that ... " we do not exclude there being more than one such object. All the existential quantifier requires is the existence of at least one. This is in keeping with the common usage of statements like (1). In general, then, if "Fx" stands for an open sentence containing "x", then "$(\exists x)(Fx)$" is true if and only if there exists at least one value for "x" that makes "Fx" true.

Now what (3) asserts can be stated in ordinary English in several ways aside from (1), for example,

> There is an object that is a building and is over 1200 feet tall
>
> Something is a building over 1200 feet tall
>
> Some building is over 1200 feet tall
>
> A building over 1200 feet tall exists.

The logical notation (3) regularizes this variety of idioms, and puts into prominence both the existential nature of what they assert and the two monadic predicates that play a role, namely, "① is a building" and "① is over 1200 feet tall".

Statement (3) may also be taken as a paraphrase of

> Some buildings are over 1200 feet tall
>
> There are buildings over 1200 feet tall
>
> Buildings over 1200 feet tall exist.

To be sure, in some settings the use of plurals may be meant to convey the existence of at least two such buildings, and hence to convey a claim stronger than (3). However, often no such stronger claim is intended; moreover, usually it is the bare existence claim (3) that is essential to the structure of logical arguments in which these statements figure. Hence we shall treat statements like these in the plural just as we

treat their counterparts in the singular. (In §41 we shall introduce notation for paraphrasing claims that there exists more than one object of a certain sort; but we reserve this notation for statements in which the requirement of several objects is made explicit.)

A large number of English statements in which occurs "exists", "there is", "there are", or "some" can be paraphrased using the existential quantifier. The statement

$$(\exists x)(x \text{ is a philosopher} \,.\, x \text{ is wise})$$

is true if and only if at least one object is both a philosopher and wise; hence it is a paraphrase of "Some philosophers are wise", "Some philosopher is wise", "There is a wise philosopher", "There are wise philosophers", and "Wise philosophers exist". Similarly, "$(\exists x)(x$ is a satellite of Jupiter)" amounts to "There exists a satellite of Jupiter", and hence also to "Jupiter possesses a satellite". Note that in the latter, the indefinite article does the work of "some". The statement

$$(\exists x)(x \text{ is a philosopher} \,.\, x \text{ loves dogs})$$

amounts to the same as "Some philosopher loves dogs", as "There is a dog-loving philosopher", and as "Some philosophers are dog lovers". Note here the variety of ways in English in which the monadic predicate "① loves dogs" may be expressed.

Here are some more complex examples. The statements "Some philosophers are not wise" and "There is an unwise philosopher" can both be paraphrased

$$(\exists x)(x \text{ is a philosopher} \,.\, -(x \text{ is wise})).$$

For the latter is true if and only if some value of "x" makes "x is a philosopher" true but makes "x is wise" false. The statements

> Some philosophers are both wise and clever
> Some wise philosophers are clever
> Wise, clever philosophers exist

can all be paraphrased

$(\exists x)(x$ is a philosopher . x is wise . x is clever$)$,

whereas

> Some philosophers are wise but not clever
> There is a philosopher who, although wise, is not clever
> Some wise philosophers fail to be clever

can be paraphrased

$(\exists x)(x$ is a philosopher . x is wise . $-(x$ is clever$))$.

"There are philosophers who are either wise or clever", or, what amounts to the same, "Some philosophers are wise or clever", can be paraphrased

$(\exists x)(x$ is a philosopher . $(x$ is wise $\vee$ x is clever$))$.

Note here the internal pair of parentheses, which are needed for grouping.

The words "something" and "someone" often go over into

existential quantifiers, although there is a difference between them. "Vanessa sees something in the garden" can be written

$(\exists x)(x$ is in the garden . Vanessa sees $x)$,

but "Vanessa sees someone in the garden" should be written

$(\exists x)(x$ is a person . x is in the garden . Vanessa sees $x)$.

"Someone", of course, amounts to "some person". By the way, the same paraphrase may be used for "Vanessa sees a person in the garden"; as in a previous example, the indefinite article has the force of an existential quantifier.

However, sometimes in the paraphrase of a statement containing "someone", the clause "x is a person" may be omitted. "$(\exists x)(x$ is in the garden . x is reading Flaubert$)$" does perfectly well for "Someone in the garden is reading Flaubert", for only persons read Flaubert, so that any value for "x" that makes "x is reading Flaubert" true will also make "x is a person" true. To include "x is a person" explicitly would add nothing. The same applies, for example, to "Someone in the garden is a philosopher" and to "Someone in Her Majesty's employ is a double agent".

Existential quantifications may themselves be compounded truth-functionally. "Some philosophers are wise and some philosophers are clever" is a conjunction of statements, and may be paraphrased

$(\exists x)(x$ is a philosopher . x is wise$)$. $(\exists x)(x$ is a philosopher . x is clever$)$.

Similarly, "Some philosophers are wise or some philosophers are clever" and "If some philosophers are wise then some philosophers are clever" are paraphrased

$(\exists x)(x \text{ is a philosopher} \,.\, x \text{ is wise}) \vee (\exists x)(x \text{ is a philosopher} \,.\, x \text{ is clever})$

$(\exists x)(x \text{ is a philosopher} \,.\, x \text{ is wise}) \supset (\exists x)(x \text{ is a philosopher} \,.\, x \text{ is clever})$,

respectively.

The open sentence enclosed in parentheses that follows an existential quantifier is called the *scope* of that quantifier. In each of the three statements just displayed, the scope of the first existential quantifier is "x is a philosopher . x is wise", and that of the second is "x is a philosopher . x is clever". Just as the scope of a negation sign "–" gives us the limits of the sentence that is being negated, the scope of an existential quantifier tells us what open sentence is being quantified. The demarcation of scope is essential for both negation signs and existential quantifiers. Just as we must distinguish between "$-(p.q)$" and "$-p.-q$", we must distinguish between

(4) $(\exists x)(x \text{ is a horse} \,.\, x \text{ has wings})$

and

(5) $(\exists x)(x \text{ is a horse}) \,.\, (\exists x)(x \text{ has wings})$.

The difference between (4) and (5) is the difference between "Something is a horse and has wings" and "Something is a horse, and something has wings". (4) is true if and only if there is an object of which "① is a horse" and "① has wings" are both true; hence (4) is true if and only if there is a winged

horse. (5) is true if and only if there is an object of which "① is a horse" is true and there is an object of which "① has wings" is true; (5) does not require that these be the same object. Thus, as it happens, (4) is false but (5) is true.

This example shows, roughly put, that occurrences of "x" that lie in the scopes of different quantifiers act like different variables. Occurrences of "x" inside the scope of a quantifier are insulated by that quantifier from the parts of the sentence outside that scope. This phenomenon is worth elaborating.

If "Fx" stands for an open sentence containing "x" and no other variables, then "$(\exists x)(Fx)$" represents a statement: it is either true or false. Thus the variable "x" no longer has the role it had in "Fx"; it no longer awaits determination by an assignment of a value. This should be clear from our explanation: "$(\exists x)(Fx)$" is true just in case there is some value for "x" that makes "Fx" true. Equally clear from this should be the fact that which variable is quantified makes no difference, as long as it is the variable in the open sentence. Thus "$(\exists x)(x$ is a philosopher . x is wise$)$" amounts to the same as "$(\exists y)(y$ is a philosopher . y is wise$)$", and, in general, "$(\exists x)(Fx)$" amounts to the same as "$(\exists y)(Fy)$".

We say that "$(\exists x)$" *binds* the variable "x"; in a quantified sentence, every occurrence of "x" in the scope of a quantifier "$(\exists x)$" is bound by that quantifier (and, for convenience, we also say that the occurrence of "x" in the quantifier "$(\exists x)$" itself is bound by the quantifier). Thus, in (5) the first two occurrences of "x" are bound by the first existential quantifier, and the second two are bound by the second quantifier. A sentence is open if it contains a *free* occurrence of a variable, that is, an occurrence not bound by any quantifier. It is only the free occurrences of a variable that can be assigned values. The role of a bound occurrence of a variable, on the other hand, is confined to the scope of the quantifier that binds it.

§20. The Universal Quantifier

Analogous to the existential quantifier is the *universal quantifier* "$(\forall x)$", which can be read "for all x", "for every x", or "every object x is such that". If "Fx" stands for an open sentence containing free "x", then "$(\forall x)(Fx)$" is true if and only if every assignment of a value to "x" makes "Fx" true. Thus

(1) $(\forall x)(x$ is animal $\vee$ x is vegetable $\vee$ x is mineral$)$

is true if and only if "x is animal $\vee$ x is vegetable $\vee$ x is mineral" is true for each value of "x". That is, (1) is true if and only if everything is either animal, vegetable, or mineral. Like an existential quantifier, a universal quantifier has a scope that is demarcated by parentheses, and the quantifier "$(\forall x)$" binds the occurrences of "x" in its scope. Thus (1) can be contrasted with

(2) $(\forall x)(x$ is animal$)$ $\vee$ $(\forall x)(x$ is vegetable$)$ $\vee$ $(\forall x)(x$ is mineral$)$,

which is true just in case either everything is animal, or everything is vegetable, or everything is mineral. A distinction in scope can be seen even more vividly in

$(\forall x)(x$ is red $\vee$ $-(x$ is red$))$

as opposed to

$(\forall x)(x$ is red$)$ $\vee$ $(\forall x)(-(x$ is red$))$.

The former amounts to "Everything is either red or not-red". It is true; indeed, it is logically true. The latter amounts to

"Either everything is red or everything is not-red" and is false: since there are things that are not red and there are things that are red, neither disjunct is true.

Common among statements that involve "all" or "every" are ones like

All philosophers are wise
All fish swim
Every pet obeys its master.

(There is no logical distinction between statements that use "all" and the plural and those that use "every" and the singular.) Let us see how to paraphrase such statements using the universal quantifier. Since "Some philosophers are wise" amounts to "$(\exists x)(x$ is a philosopher $\cdot$ x is wise$)$", it may be tempting to think that "All philosophers are wise" could be written "$(\forall x)(x$ is a philosopher $\cdot$ x is wise$)$". This is completely wrong, however—and shows how misleading a superficial grammatical similarity can be. "$(\forall x)(x$ is a philosopher $\cdot$ x is wise$)$" is true if and only if every object both is a philosopher and is wise, which is clearly not what "All philosophers are wise" means. Rather, "All philosophers are wise" can be rephrased, inelegantly but suggestively, as "Everything that is a philosopher is wise"; the latter, in turn, can be expressed (yet more inelegantly) as "Every object is such that if it is a philosopher then it is wise". This last is easily transcribed

(3) $(\forall x)(x$ is a philosopher $\supset$ x is wise$)$.

To check the accuracy of this paraphrase, note that (3) is true if and only if every value for "x" makes "x is a philosopher $\supset$ x is wise" true. Hence it is true if and only if no value for

"x" makes "x is a philosopher $\supset$ x is wise" false. By truth-functional logic, a value for "x" makes this open sentence false if and only if it makes "x is a philosopher" true and makes "x is wise" false; this occurs just in case that value is a philosopher but is not wise. Thus (3) is true if and only if no individual is a philosopher but is not wise. This is just the condition under which "All philosophers are wise" is true, and our paraphrase is vindicated.

Similarly, the other statements displayed above can be paraphrased

$(\forall x)(x$ is a fish $\supset x$ swims$)$

$(\forall x)(x$ is a pet $\supset x$ obeys x's master$)$,

respectively. Of the same form are "Everything that breathes has lungs" and "Clive bought everything he saw", which are paraphrased "$(\forall x)(x$ breathes $\supset x$ has lungs$)$" and "$(\forall x)($Clive saw $x \supset$ Clive bought $x)$". A variety of linguistic forms not explictly containing "all" or "every" can be used to the same effect. The statements

Honorable people pay their debts

The honorable person pays his/her debts

An honorable person pays his/her debts

all mean the same as "All honorable people pay their debts", and hence are all paraphrased "$(\forall x)(x$ is an honorable person $\supset x$ pays x's debts$)$". Note that the last of these statements uses the indefinite article as a universal quantifier. Other examples of this are "A Boy Scout is thrifty" and "A double agent is a dangerous person". This is in contrast to the use of the indefinite article as an existential quantifier, a

use we saw in the previous section and can see in "A double agent is in Her Majesty's employ".

A locution like "Only alumni are eligible" also amounts to a universally quantified conditional, but care must be taken as to which monadic predicate occurs in the antecedent and which in the consequent. This statement means the same as "An individual is eligible *only if* that individual is an alumnus", and hence is paraphrased

(4) $(\forall x)(x$ is eligible $\supset x$ is an alumnus$)$.

Thus the statement must be distinguished from "All alumni are eligible". Note that (4) is also a correct paraphrase of "None but alumni are eligible".

Universally quantified conditionals have the following property: if no value for "x" makes the antecedent of the conditional true, then the universally quantified conditional is true. For if no value for "x" makes the antecedent true, then every value for "x" makes it false; whence every value for "x" makes the conditional true, so that the universal quantification is true. As a result, suppose we take "All my roommates are loathsome" to mean

$(\forall x)(x$ is a roommate of mine $\supset x$ is loathsome$)$.

It follows that if, in fact, I have no roommates, then the statement comes out true. This may seem odd. The oddity arises, I think, because "All my roommates are loathsome" can, in some conversational settings, be meant to convey "I have roommates and all of them are loathsome", that is,

$(\exists x)(x$ is a roommate of mine$)$. $(\forall x)(x$ is a roommate of mine $\supset x$ is loathsome$)$.

Ordinary usage here is somewhat ambiguous. Whether, and in what circumstances, such ordinary language "all"-statements carry *existential import*—require for their truth the existence of a value for "x" that makes the antecedent of the conditional true—is beside the point for us. What *is* important is that quantificational notation can exhibit the difference between the two interpretations of such statements. Below we shall always use such "all"-statements without existential import; we shall interpret them as simple universal quantifications.

Slightly more complex conditionals can be used in paraphrases of more complex English statements. Consider, for example,

All philosophers are either wise or clever
All philosophers are wise and clever
All philosophers who read Frege are clever
All wise philosophers are clever.

For brevity, let us use "Px" for "x is a philosopher", "Wx" for "x is wise", "Cx" for "x is clever", and "Fx" for x reads Frege". Then the four statements may be paraphrased

$$(\forall x)(Px \supset Wx \vee Cx)$$
$$(\forall x)(Px \supset Wx . Cx)$$
$$(\forall x)(Px . Fx \supset Cx)$$
$$(\forall x)(Px . Wx \supset Cx).$$

The second and fourth of these provide another dissimilarity to existential statements that is masked by grammatical similarity. We saw in §19 that "Some philosophers are wise and clever" and "Some wise philosophers are clever" are

both paraphrased "$(\exists x)(Px . Wx . Cx)$"; they both require the existence of at least one individual that is a philosopher, is wise, and is clever. But "All philosophers are wise and clever" and "All wise philosophers are clever" are different, as is shown by their paraphrases. The former requires of each object, that if it is a philosopher then it is wise and clever; the latter requires, of each object, that if it is both a philosopher and wise then it is clever.

The word "everyone" means the same as "every person", and hence indicates a universal quantifier. Thus "Everyone in the garden is cold" amounts to "$(\forall x)$(x is a person . x is in the garden $\supset$ x is cold)". The clause "x is a person" is essential here; were it omitted, the result would amount to "Everything in the garden is cold". But, as with "someone", the clause "x is a person" can be omitted when it adds nothing, as in "Everyone who enjoys Flaubert detests Zola", which can be accurately paraphrased "$(\forall x)$(x enjoys Flaubert $\supset$ x detests Zola)".

Universal quantifications may be truth-functionally compounded, and may be truth-functionally compounded with existential quantifications. "If every philosopher reads Frege then every philosopher is clever" and "Either every philosopher reads Frege or there is an unwise philosopher" are paraphrased

$$(\forall x)(Px \supset Fx) \supset (\forall x)(Px \supset Cx)$$
$$(\forall x)(Px \supset Fx) \vee (\exists x)(Px . -Wx),$$

respectively, where we use the abbreviations introduced above.

Although the paraphrase of statements like those we have seen should become reasonably automatic with practice, more complex statements may require some thought. In such cases, as in truth-functional paraphrase, it helps to paraphrase inward. The principal new step comes in dealing with quantifications. To treat a universal quantification, it helps to put it in the form "Every object x is such that ... ", where " ... " represents an open sentence. The first task, then, is to formulate this open sentence, using free "x". After that, the open sentence may itself be analyzed truth-functionally. For example, to paraphrase

> Every student who takes logic and reads Frege or Russell will pass the examination and will, if s/he works hard, gain an excellent background

we might start by rewriting

> Every object x is such that: if x is a student who takes logic and reads Frege or Russell, then x will pass the examination and, if x works hard, then x will gain an excellent background.

The open sentence following the colon is, evidently, a conditional. Its antecedent can be rephrased

> x is a student . x takes logic . (x reads Frege ∨ x reads Russell).

For the rest, the truth-functional analysis is straightforward. The resulting paraphrase of the whole statement is

> $(\forall x)$[x is a student . x takes logic . (x reads Frege ∨ x reads Russell) ⊃ x passes the examination . (x works hard ⊃ x gains an excellent background)]

or, using "Sx", "Lx", and so on, as abbreviations for the constituent open sentences,

$$(\forall x)[Sx.Lx.(Fx \vee Rx) \supset Ex.(Hx \supset Gx)].$$

Statements that contain "nothing", "no one", or "no" can often be paraphrased quantificationally. Consider

> No philosopher is wise
> Nothing in the shop is worth buying
> No one in the room is awake.

The first of these is true if and only if every individual that is a philosopher fails to be wise; hence it may be paraphrased

> $(\forall x)$(x is a philosopher ⊃ –(x is wise)).

Similarly, the other two may be paraphrased

> $(\forall x)$(x is in the shop ⊃ –(x is worth buying))
> $(\forall x)$(x is a person . x is in the room ⊃ –(x is awake)).

There are alternatives here. "No philosopher is wise" is true just in case there is no individual that is a philosopher and is wise. Thus, "No philosopher is wise" amounts to "It is not the case that there is a wise philosopher". Thus, as an alternative paraphrase, we can use

> –$(\exists x)$(x is in the shop . x is worth buying)
> –$(\exists x)$(x is a person . x is in the room . x is awake).

The equivalence of the negation of an existential quantification to a universal quantification is, in fact, a logical law. More precisely, if "Fx" represents any open sentence,

> "$-(\exists x)Fx$" is true if and only if "$(\forall x)-Fx$" is true

"–(∀x)Fx" is true if and only if "(∃x)–Fx" is true.

(Here and henceforth we omit the parentheses surrounding the scope of a quantifier if the scope is represented by a simple expression like "Fx" or "–Fx".) For "–(∃x)Fx" is true if and only if no value for "x" makes "Fx" true. This occurs just in case every value for "x" makes "Fx" false, that is, just in case "(∀x)–Fx" is true. Similarly, "–(∀x)Fx" is true if and only if not every value for "x" makes "Fx" true. This occurs just in case some value for "x" makes "Fx" false, that is, just in case "(∃x)–Fx" is true. Hence

–(∃x)(x is a philosopher . x is wise)

is true if and only if

(∀x)(–(x is a philosopher . x is wise))

is true. The scope of "(∀x)" here has the form "–(p.q)", which is truth-functionally equivalent to "$p \supset -q$". Hence the latter statement is true if and only if

(∀x)(x is a philosopher ⊃ –(x is wise))

is true, as desired.

Thus the negation of "Some philosophers are wise" amounts to "No philosopher is wise", that is, to "All philosophers are unwise". In like manner, the negation of "Some philosophers are not wise" amounts to "All philosophers are wise".

Our logical law tells us, moreover, that "(∃x)Fx" is true if and only if "–(∀x)–Fx" is true, and that "(∀x)Fx" is true if and only if "–(∃x)–Fx" is true. Thus we can eliminate the existential quantifier in favor of the universal quantifier, by replacing each "(∃x)" with "–(∀x)–". Alternatively, we can eliminate the universal quantifier in favor of the existential, by replacing each "(∀x)" with "–(∃x)–". However, for convenience and readability we shall continue to use both sorts of quantifier.

§21. Further Notes on Paraphrase

Paraphrase of ordinary language into quantificational notation is sometimes a subtle matter, particularly in the demarcation of the scopes of the quantifiers. There are few general rules for this. One simply has to rethink, in quantificational terms, what the statements under consideration are meant to convey. In this section we briefly and unsystematically sample some of the problems that can arise.

In statements that contain both a quantifier and a negation, it is not always clear whether the negation lies within the scope of the quantifier or rather whether it negates a whole quantified statement. To be sure, the statement "Shaw does not like some Wagner operas" amounts to "There is some Wagner opera that Shaw does not like", and so can be rendered

(1) (∃x)(Wx.–Lx),

where "Wx" stands for "x is a Wagner opera" and "Lx" for "Shaw likes x". Moreover, the statement "There isn't a Wagner opera that Shaw likes" can be symbolized

(2) –(∃x)(Wx.Lx).