Philosophy 230

Wesleyan University
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Handout 5a

General Laws of Quantificational Logic

I. General Laws: *Truth-Functional Laws*

A. We now turn to general laws connecting validity, satisfiability, implication and quantificational schemata.

B. Let's first note that since the truth-functional connectives are treated in exactly the same way in quantificational interpretations as in truth-functional ones, all the basic facts concerning the validity, satisfiability, implication, and so forth of truth-functional schemata continue to hold for quantificational schemata. These then are also logical laws governing quantificational schemata.

C. So, for example, a conjunction of quantificational schemata is valid iff, each of the conjuncts is valid. For, if $A.B$ is valid, and we are given an arbitrary interpretation I, then $I \vDash A.B$. Hence $I \vDash A$ and $I \vDash B$. But I is an arbitrary interpretation, so the foregoing reasoning holds for all intepretations. So A and B are both valid..

II. General Laws: *Substitution*

A. The law of substitution in the TF case states that whenever we substitute a schema B for each occurence of some sentence letter in a given schema A, the resulting schema will be valid if A is and unsatisfiable if A is. Moreover, if C results from substituting B for a sentence letter in A, and C' results from substituting B' for a sentence letter in A', $A \Rightarrow A'$ then $C \Rightarrow C'$.

B. In the quantificational case this has to be modified because some of the *parts* of quantificational schemata are not sentence letters but *predicate* letters.

C. So the question is, what can we substitute for a predicate letter? The answer is: a *complex schematic predicate.*

D. For example, we may substitute

$$F①\supset G①$$

for

$$F①$$

in

$$-(\forall x)(Fx)$$

to get:

$$-(\forall x)(Fx \supset Gx)$$

E. In general, a complex schematic predicate is obtained from an open quantificational schema (i.e., one containing free variables) by replacing some free variables with placeholders.

F. The Law of Substitution then is

   Substitution of schemata for sentence letters and complex schematic predicates for predicate letters preserves validity, implication, satisfiablity and equivalence.

G. The law of substitution for quantificational quantificational logic implies that, since $-(\forall x)(Fx)$ is equivalent to $(\exists x)(-Fx)$,

$$-(\forall x)(Fx \supset Gx)$$

   is equivalent to

$$(\exists x) - (Fx \supset Gx)$$

III. Substitution Restrictions.

A. There are three conditions that have to be satisfied for a substitution to be legitimate.

B. Firstly, any variable free in the schema replacing a sentence letter must *not become bound*, that is, be captured by a quantifier.

   1. So, for example, we have seen that
   $$p \supset (\forall x)Fx \Leftrightarrow (\forall x)(p \supset Fx);$$
   By the Law of Substitution, we may substitute Gy for p to obtain:
   $$Gy \supset (\forall x)Fx \Leftrightarrow (\forall x)(Gy \supset Fx);$$
   2. But, we may **NOT** substitute Gx for p to obtain:
   $$Gx \supset (\forall x)Fx \Leftrightarrow (\forall x)(Gx \supset Fx);$$
   because x after the substitution becomes bound by $(\forall x)$. Indeed it should be obvious that the left schema is true if something is not F, provided that it also is not G; but clearly in this case the consequent of the left-hand side is false.

C. Secondly, if we substitute a predicate for some predicate letter, the variables which take the place of the placeholders in the predicate must not be captured by quantifiers internal to the predicate.

   1. For example, if we have the schema
   $$(\forall x)(Fxz \supset Fxz \vee Gx),$$
   2. we may replace
   $$F①② \text{ by } (\exists y)R①②y,$$
   to get the schema
   $$(\forall x)[(\exists y)Rxzy \supset (\exists y)Rxzy \vee Gx]$$
   3. But we may not replace
   $$F①② \text{ by } (\exists z)R①②z,$$
   since the z entering the placeholder ② will become bound by the quantifier.

D. Thirdly, if there are any free variables in the schematic predicate, these must not become bound by any quantifier:

   1. So, for example, in the schema

      $$(\forall x)[Fxy \supset (\exists z)Fxz],$$

      we may replace

      $$F①② \text{ by } G①②y,$$

      getting:

      $$(\forall x)[Gxyy \supset (\exists z)Gxyz].$$

   2. But we may not replace

      $$F①② \text{ by } G①②z,$$

      since the 'z' would then be captured by the existential quantifier, viz.,

      $$(\forall x)(Gxyz \supset (\exists z)Gxzz),$$

      which is not valid.

IV. Proof of the Substitution Law

A. How do we prove this law? Let us just think about the validity part. Suppose that we have some valid quantificational schema.

B. Then, no matter what the extensions of the predicate letters, the universe of discourse, and the assignments of objects to the free variables, the schema must come out $\top$.

C. So, suppose we substitute some quantificational open sentence for a predicate letter, $F①$, and consider some arbitrary interpretation.

D. The quantificational open sentence will be $\top$ of the objects in some set, under this interpretation; so consider the interpretation which assigns this set to F.

E. It is not difficult to see that, since the original schema must be $\top$ under this interpretation, so must our new schema.

F. The proof of this last statement is itself not difficult, conceptually, but it is tedious.

   1. What we are claiming is that, if A is a schema and B is a substitution instance of A, then,
   2. given any interpretation I such that $I \vDash B$,
   3. there is an interpretation $I' \vDash A$ such that
   4. $I' \vDash A$ if and only if $I \vDash B$; i.e., A has the same truth-value under I' as B has under I.

G. To prove this rigorously, we have to consider **all** schemata, and therefore must proceed *by mathematical induction*. You will be relieved to hear that we we will not discuss it here.

V. General Laws: *Interchange*

A. The law of interchange, in the TF case, states that, if we replace any schema which occurs in a TF schema by an equivalent schema, the resulting schema will be equivalent to the original schema; moreover, interchange of equivalent schemata preserves validity, unsatisfiability, implication, and even satisfiability.

B. In the case of quantificational schemata, we must again re-formulate the law of interchange.

C. The law of interchange states that we may replace any quantificational schema which is *part of* a larger schema by any equivalent *quantificational* schema and that the resulting schema will be equivalent to the original.

D. For example, if we start with the schema:

$$(\forall x)(Fx \supset Gx) \supset (\exists x)(Fx \vee Gx)$$

we may replace

$$Fx \supset Gx$$

with the equivalent schema

$$-Fx \vee Gx$$

to get:

$$(\forall x)(-Fx \vee Gx) \supset (\exists x)(Fx \vee Gx)$$

The law of interchange for quantificational schemata implies that this second schema is equivalent to the first.

E. Similarly, if we have the schema:

$$(\exists x) - (Fx \supset Gx)$$

we may replace

$$-(Fx \supset Gx)$$

by

$$Fx. - Gx$$

to get:

$$(\exists x)(Fx. - Gx)$$

Once more, the law of interchange implies that these schemata are equivalent.

F. Note that since

$$(\exists x) - (Fx \supset Gx)$$

is equivalent to:

$$-(\forall x)(Fx \supset Gx)$$

the last example shows that these schemata are equivalent to

$$(\exists x)(Fx.-Gx)$$

G. The proof of the law of interchange is in the book. I leave it to you to read it.

VI. Simple Purely Quantificational General Laws

There are some laws, concerning quantificational schema, which have no counterparts in TF logic, since they concern the quantifiers. We have already seen a few of these, namely, the laws of distribution and the laws relating universal and existential quantification.

A. The first is called the **Law of Universal Instantiation**:

A universally quantified schema implies each of its instances

Or, more formally:

$(\forall u)\Phi(u)$ implies $\Phi(v)$

B. The second is called the **Law of Existential Generalization**:

An existentially quantified schema is implied by each of its instances

Or, more formally:

$\Phi(v)$ implies $(\exists u)\Phi(u)$

(Note that we may also choose different bound variables, other than x.)

C. The proofs of both are easy. Here's the one for Existential Generalization

1. Suppose that we have an interpretation under which Fx is $\top$.
2. Then the object assigned to x is in the extension of F.
3. But, the object assigned to x is in the UD; so, *some* object in the UD must be in the extension of F.
4. Hence, $(\exists x)Fx$ must be $\top$ under this interpretation.
5. Hence, $(\exists x)Fx$ must be $\top$ under any interpretation under which Fx is $\top$; hence, Fx implies $(\exists x)Fx$.

D. Putting these two laws together with the law of substitution, we have:

Every (open) schema is implied by its universal quantification, i.e.,

For any u and any $\Phi(u)$, $(\forall u)\Phi(u)$ implies $\Phi(u)$.

And:

Every (open) schema implies its existential quantification, i.e.,

For any u and any $\Phi(u)$, $\Phi(u)$ implies $(\exists u)\Phi(u)$.

For, every open schema is a substitution instance of Fx.

E. In general, we cannot reverse the laws of Universal Instantiation and Existential Generalization.

F. That is, a universally quantified schema is *not*, in general, implied by each of its instances, and an existential schema does not imply each of its instances.

G. The reason is that we may have an interpretation under which Fx is $\top$, but under which $(\forall x)Fx$ is not $\top$; and we may have an interpretation under which $(\exists x)Fx$ is $\top$, but under which Fx is not $\top$.

VII. The Universal Closure Law.

A. Although $\Phi(y)$ does not imply $(\forall x)\Phi(x)$, if $\Phi(y)$ is a *valid* schema, then $(\forall x)\Phi(x)$ must also be valid.

B. Here is the proof of this fact.

1. Suppose that $\Phi(x)$ is a valid schema.
2. Fix some DQ and some interpretation of the predicate letters. Since $\Phi(x)$ is valid, no matter what we assign to x, the schema must come out $\top$.
3. So, under any interpretation, with that domain and that assignment of extensions to the predicate letters, the open schema $\Phi(x)$ must be $\top$ of *every* object in the DQ.
4. Hence, since the DQ and interpretation of the predicate letters was arbitrary, under *every* choice of DQ and *every* interpretation of the predicate letters, $\Phi(x)$ is $\top$ of *every* object in the DQ.
5. But that is just to say that, under *every* interpretation, $(\forall x)\Phi(x)$ is $\top$; i.e., it's valid.

C. What this shows is that, *as far as validity is concerned*, a free variable acts just like a universally quantified variable. But, to repeat, $\Phi(y)$ does not imply $(\forall x)\Phi(x)$, so there is quite a difference between an open schema and its universal quantification.

D. This last fact can be applied repeatedly. If

$$\Phi(x_1, \ldots, x_n)$$

is valid, then so is

$$(\forall x_n)\Phi(x_1, \ldots, x_n)$$

similarly, so is

$$(\forall x_{n-1})(\forall x_n)\Phi(x_1, \ldots, x_n)$$

and, in the end, we shall have that

$$(\forall x_1)(\forall x_2) \ldots (\forall x_n)\Phi(x_1, \ldots, x_n)$$

is valid.

E. So, in general, if a schema containing some number of free variables is valid, then the schema consisting of that schema, prefixed by universal quantifiers which bind those free variables, is also valid. We call this latter schema the *universal closure* of the original schema. Note that it contains no free variables.

VIII. Rules of Inference

A. Notice that each of the Laws that we have just quickly gone over tells us that any schema of a certain form implies some corresponding schemata of another form.

B. For example, Universal Instantiation tells us that any schema of the form, $(\forall u)\Phi(u)$ implies a schema of the form $\Phi(v)$ which is an instance of the former a schema.

C. The terminology we will adopt for this phenomenon is that there is a valid **RULE OF INFERENCE** from any schema that has the form of a universal quantification to any other schema that is its instance.

D. The motivation for this terminology should be clear. Given the Law of Universal Instantiation, we know that one can validly infer any instance from its universal quantification.

E. Clearly, the Law of Existential Generalization also yields a valid rule of inference.