Philosophy 230

Wesleyan University
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Handout 12a

Quantificational Compactness; Decision Procedures;
the Löwenheim-Skolem Theorem

I. Corollary of completeness: unsatisfiability equivalent to truth functional unsatisfiability in sequents

A. We move next to two simple corollaries of our completeness proof.

B. First, remember how I described the intuitive idea underlying completeness:

> If a generalization is self-contradictory, then some of its consequences about particular things will turn out to be mutually inconsistent.

C. The first simple corollary of completeness is that the converse of this idea is also true. I.e.,

> If there are quantifier-free sequents of $R$ whose conjunction is truth functionally unsatisfiable, then $R$ is itself quantificationally unsatisfiable.

D. The argument is very simple. Indeed, it should be already evident to you if you've thought a moment about the proof of completeness from the Central Lemma.

E. Suppose that there are quantifier free sequents of $R$ whose conjunction is truth functionally unsatisfiable.

F. Recall that the argument of this proof includes the following sub-argument: if there is a deduction from $R$ of truth functionally unsatisfiable sequents, using only the $UI$ and ***strict*** $EII$, then there is a deduction of $-R$ from no premises.

G. Now, since we know our system of deduction is sound, it follows that $-R$ is valid; but then $R$ is unsatisfiable.

II. Corollary of corollary: search procedure for unsatisfiability and validity

A. The second simple corollary follows from the first.

B. The first corollary assures us that if a schema is unsatisfiable, then application of the Rigid Plan will produce from it a truth functional contradiction.

C. This yields a *search procedure* for quantificational unsatisfiablity.

1. Given any quantificational schema, apply the Rigid Plan to it.
2. If at some point the sequents generated yield a truth functionally unsatisfiable conjunction, then we have discovered that the original schema is unsatisfiable.

D. Moreover, the Central Lemma assures us that this procedure will not lead us astray: if we obtain a truth functional contradiction, the original quantificational schema is guaranteed to be unsatisfiable.

III. The Rigid Plan doesn't yield a decision procedure

A. Note that a search procedure for unsatisfiability is also a search procedure for validity. If a schema is valid, its negation is unsatisfiable; so procedure is to generate sequents from the prenex equivalent of its negation.

B. Notice, however, that if a schema is ***satisfiable***, i.e., ***NOT UNSATISFIABLE***, this procedure will not necessarily tell us that it is.

C. There are two reasons for this.

   1. First, all we know about satisfiability on the basis of the Rigid Plan is: if a schema ***is satisfiable***, then the Rigid Plan won't produce a truth functional contradiction.
   2. Second, as we have seen, the Rigid Plan doesn't always finish after a finite number of steps.

D. Consider a schema on which the Plan doesn't terminate. At any stage there may be no truth-functional unsatisfiability yet, but we can be sure that we would not generate one by continuing further.

E. So, on a given *satisfiable* schema, if the Rigid Plan doesn't terminate, we won't ever find out that no contradiction appears.

F. The upshot of all this is that using the Rigid Plan alone, we can't, in general, decide, in a finite number of steps, whether a schema is unsatisfiable or not. The Plan does not give us a *search procedure* for unsatisfiability.

G. What is required for a decision procedure for unsatisfiability? It would have to go beyond what we have, namely, a procedure which tells us in a finite number of steps that the schema is unsatisfiable if it is. In addition, it would have to tell us in a finite number of steps that the schema in question is not unsatisfiable if it is not unsatisfiable. Put in another way, if we had *a search procedure for satisfiability* as well, we would have a decision procedure for unsatisfiability, *and* satisfiablity.

H. One final point. All of the foregoing holds for schemata on which the Rigid Plan does *not* terminate. If the Rigid Plan *does* terminate, clearly we do have a search procedure for satisfiability as well.

I. As Goldfarb notes, we can show that there are various types of quantificational schemata on which the Plan terminates. I'll talk more about these later.

IV. Quantificational Compactness

A. Compactness holds also for quantificational logic. Our next topic is a sketch of the proof of this theorem:

   > If every conjunction of members of an infinite set of quantificational schemata is satisfiable, the whole set is.

B. The argument begins with a series of slight modifications to the Rigid Plan.

C. It is fairly obvious that the Rigid Plan can be applied to finite sets of prenex schemata. Given such a set $\{Y_1, \ldots, Y_k\}$, we simply apply the Rigid Plan to all the members at stage one, and then, at each subsequent stage, we apply the Plan exactly as before, to all the sequents generate up through the stage at which we have arrived.

D. Now, what do we do with this extension of the Rigid Plan. Well, it is clear that we can prove a form of the Central Lemma for finite sets of quantificational schemata:

If a set $\{Y_1, \ldots, Y_k\}$ is not jointly satisfiable, then we can deduce from them, by $UI$ and strict $EII$, a finite number of quantifier free schemata whose conjunction is truth functionally unsatisfiable.

E. The proof is little more than a repetition of the original proof.

F. If we never obtain quantifier-free schemata whose conjunction is truth-functionally unsatisfiable, then (applying, if necessary, Truth-functional Compactness) we can define an interpretation under which all of $Y_1, \ldots, Y_k$ are true, that is, under which the conjunction $Y_1 \, . \, \ldots \, . \, Y_k$ is true.

G. If, on the other hand, we do generate quantifier-free schemata whose conjunction is unsatisfiable, then the negation of $Y_1 \, . \, \ldots \, . \, Y_k$ is deducible, whence by soundness the $Y_1 \, . \, \ldots \, . \, Y_k$ is unsatisfiable, and so $Y_1, \ldots, Y_k$ are not jointly satisfiable.

H. Now, let's see how we can modify the Rigid Plan so as to apply to infinite sets of prenex schemata. Let $\{Y_1, Y_2, \ldots\}$ be such a set. The modification needed is as follows.

1. At stage 1 we write down $Y_1$.
2. At each subsequent stage $n+1$, we first apply the standard Rigid Plan to generate new sequents from those obtained up through stage $n$.
3. Then, we also write down $Y_{n+1}$.

I. Thus, at stages after $n+1$, further sequents of $Y_1, \ldots, Y_{n+1}$ will be generated, as well as more of the schemata $Y_k$, $k > n+1$. Let's call this procedure the infinitary Rigid Plan.

J. Once we provide a precise statement of the procedure as described roughly above, we prove a result very much like the Central Lemma, using, indeed an argument very much like the argument of the Central Lemma. Here is the theorem:

Either
(a) at some stage in the application of the infinitary Rigid Plan quantifier-free sequents will be generated whose conjunction is truth-functionally unsatisfiable; or
(b) there is an interpretation under which all of $Y_1$, $Y_2$, ... are true.

K. Now, if part (a) of the theorem holds for a given infinite set of schemata, and if $k$ is an integer such that there exists a truth-functionally unsatisfiable conjunction of sequents of $Y_1, \ldots, Y_k$, then the negation of $Y_1 \, . \, \ldots \, . \, Y_k$ is deducible, so that $Y_1 \, . \, \ldots \, . \, Y_k$ is unsatisfiable.

L. It follows that if *every* conjunction of members of $\{Y_1, Y_2, \ldots\}$ is satisfiable, then (a) cannot hold.

M. But the assumption of the Compactness Theorem is that *every* conjunction of members of $\{Y_1, Y_2, \ldots\}$ is satisfiable, so it follows that (b) holds. That is, there is an interpretation under which all of $Y_1$, $Y_2$, ... are true, that is, the set $\{Y_1, Y_2, ...\}$ is satisfiable. This concludes the proof of Quantificational Compactness.

V. Löwenheim-Skolem Theorem

The It is also worth noting that the proof we have just given establishes the following theorem. The Theorem is a weak version of the so-called Löwenheim-Skolem Theorem: If a sentence $S$ is satisfiable, then it is satisfiable in some interpretation whose universe of discourse consists only of natural numbers.