Philosophy 230

Wesleyan University
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Handout 11c

Completeness II: Proof of the Central Lemma,
Truth-Functional Compactness

I. Possible Outcomes of the Rigid Plan

A. The Plan terminates with a set of quantifier-free sequents that is unsatisfiable. Our example is $(\forall x)(\forall y)(\exists z)(Fxz. - Fyz)$

B. The Plan does not terminate. Our example is $(\forall x)(\exists y)Fxy$

C. One final possibility we have not yet seen: the Plan terminates but the set of quantifier-free sequents is satisfiable. Here is an example:

1. $(\exists x)(\forall y)Fxy$
2. $(\forall y)Fa_1y$
3. Fa_1a_1

II. Proof of the Central Lemma: defining the verifying interpretation

A. Let Σ be the set of all quantifier-free sequents generated during the execution of the Rigid Plan, starting with R.

B. By the hypothesis of the Central Lemma,no set of sequents whose conjunction is TF-unsatisfiable is generated, so every conjunction of sequents in the set Σ is TF-satisfiable.

C. Now, if the Rigid Plan, applied to R, terminates, then we have possibility 3 above, and Σ is a finite set of schemata, so that the conjunction of all elements of Σ is a TF-satisfiable schema. It clearly follows that Σ itself is TF-satisfiable.

D. If the Rigid Plan does not terminate on R, we need to appeal to a theorem about truth functional logic that we will not prove until later. The theorem is called the compactness theorem of TF logic:

> Suppose that Σ is an *infinite* set of truth-functional schemata, and suppose that every (*finite*) conjunction of members of Σ is satisfiable. Then the set Σ is itself satisfiable.

E. It follows from the compactness theorem for TF-logic that if our set of sequents, Σ is infinite, it is TF-satisfiable.

F. Since Σ is truth-functionally satisfiable, there is some assignment of truth values to the atomic polyadic schemata that occur in members of Σ such that all of the elements of Σ come out true.

G. These atomic polyadic schemata are predicate letters followed by some appropriate number of the variables a_1, a_2, and so forth. Call this assignment of truth values to the atomic schemata occurring in Σ the assignment A.

H. Using the assignment A, we construct a polyadic interpretation of all sequents of Σ. This interpretation, call it I, like all polyadic interpretations, has three parts: a universe of discourse; an assignment of a member of the universe of discourse to each free variable a_i; and an assignment of an appropriate extension to each of the predicate letters.

I. The domain of I is defined as follows. For each variable a_i which occurs in any sequent of Σ, we put the natural number i itself into the DQ of I. More precisely, $DQ = \{i \in \mathbb{N} : a_i \text{ occurs in a sequent of } \Sigma\}$ (Thus, the universe of discourse is some initial segment of the natural numbers.)

J. The extension of each predicate letter is defined as follows: Each predicate letter is true of exactly those numbers, or pairs of numbers, or triples of numbers, and so forth, that the assignment A says it is true of. More precisely, if P is a predicate letter that occurs in Σ, then $extP = \{< i_1, \ldots, i_n >: Pa_{i_1}, \ldots, a_{i_n} \text{ is assigned } \top \text{ by } A\}$

   1. So, for example, if $Fa_9a_5a_{16}$ is a member of Σ, and A assigns $\top$ to $Fa_9a_5a_{16}$, then the ordered triple $< 9, 5, 16 >$ is in the extension of F①②③, i.e., $< 9, 5, 16 > \in extF$.
   2. If A assigns $\perp$ to $Fa_9a_5a_{16}$, then keep the ordered triple $< 9, 5, 16 >$ out of the extension of F①②③, i.e., $< 9, 5, 16 > \notin extF$.

K. Finally, each free variable a_i occurring in Σ is assigned the natural number i, i.e., $a_i := i$.

III. Proof of the Central Lemma: showing that the interpretation verifies

A. Now we come to the hard part of the proof. We have to show that in fact $I \vDash R$.

B. What we shall prove is that for every sequent S which occurs at any stage of the execution of the Rigid Plan on R, $I \vDash S$. Since the schema with which we started, R, is the first of these S, $I \vDash R$, hence R is satisfiable.

C. The argument strategy is the familiar inductive one. We establish that, for any number n, if I makes all sequents with n quantifiers $\top$, then it makes all sequents with $n + 1$ quantifiers $\top$. If this conditional claim is true, then, since I does, by hypothesis, make all quantifier-free sequents, i.e., sequents with 0 quantifiers, true, it makes all sequents true.

D. The antecedent of the conditional claim is, as always, the induction hypothesis; that is, in the following argument we will assume that I makes all sequents with n quantifiers $\top$.

E. Suppose now that S is a schema which contains $n + 1$ quantifiers. There are just two cases to consider.

F. First, S might be an existential schema, that is, a schema of the form $(\exists x)\Phi(x)$.

   1. By definition of the rigid plan, there is some schema $\Phi(a_i)$ which occurs during its execution (indeed, at the stage immediately following the introduction of this schema).
   2. Now, $\Phi(a_i)$ contains only n quantifiers, so by the induction hypothesis $I \vDash \Phi(a_i)$.
   3. But then, when we assign i to x, $\Phi(x)$ comes out $\top$.
   4. Hence, $I \vDash (\exists x)\Phi(x)$; that is, $I \vDash S$.

G. The second possibility is that S is a universal quantification, i.e., it is of the form $(\forall x)\Phi(x)$.

1. Let m be an arbitrary member of the universe of discourse. We must show that $\Phi(x)$ is $\top$ if we assign m to x.
2. By definition of I, m is assigned to the free variable a_m; and, since m is in the universe of discourse, a_m must occur in some sequent.
3. Now, by definition of the Rigid Plan, since the variable a_m occurs in some sequent, we must at some stage have instantiated the schema $(\forall x)\Phi(x)$ with it.
4. That is, we must, at some stage, have written down $\Phi(a_m)$.
5. But this schema contains only n quantifiers, so by the induction hypothesis $I \vDash \Phi(a_m)$.
6. It follows that $\Phi(x)$ is true when $x := m$. Since m was arbitrary, we conclude that, no matter what we assign to x, $\Phi(x)$ comes out $\top$. That is, $I \vDash (\forall x)\Phi(x)$.

H. This completes the argument for showing that I makes all sequents in the execution of the Rigid Plan for R true. Hence, $I \vDash R$, and so we have *almost* completed the proof of the Central Lemma.

I. The only missing piece is a proof of the compactness of truth functional logic.

IV. Truth functional Compactness

A. The strategy of proof of truth functional compactness is in many ways very similar to that of the Central Lemma. Here also we will first specify a procedure for generating schemata, just as for the Central Lemma we had the Rigid Plan.

B. Next, we will specify an interpretation, a truth functional one in this case, that is read off the schemata generated by the procedure.

C. Finally, we will show that the interpretation thus defined does what we want it to do; in the present case, we will show that it verifies every member of S.

V. Schema generation

A. Before defining the schemata-generating procedure, we need a few definitions.

1. First let's suppose that we list the sentence letters occurring in members of S in some way, and name these sentence letters by using the list: p_1 is the first sentence letter on the list, p_2 the second, p_3 the third, and so on. In effect we are giving serial numbers to the sentence letters that occur in S. Since S contains infinitely many schemata, there may or may not infinitely many sentence letters occurring in S.
2. Call a schema X S-compatible iff $W.X$ is satisfiable for every conjunction W of members of S.
3. An example might help. Let

$$S = \{p, q, p \vee q, (p \vee q) \vee p, ((p \vee q) \vee p) \vee q, (((p \vee q) \vee p) \vee q) \vee p, \dots\}$$

   Then, clearly, p is S-compatible, since in any interpretation in which $p := \top$, every member of S is $\top$, and so every conjunction of members of S is $\top$, and so $W.p$ is $\top$ for every conjunction W of members of S.

B. We will use the following general law of truth functional logic: if conjunctions $A.C$ and $B.-C$ are both unsatisfiable, then $A.B$ is also unsatisfiable.

C. Here is how the schemata we need are generated:

1. The first schema, call it X_1, is p_1 if p_1 is S-compatible, and $-p_1$ if p_1 is not S-compatible.
2. The subsequent schemata are generated by adding a sentence letter as a conjunct if the addition continues to be compatible with S, and adding the negation of that letter otherwise.
3. More exactly, suppose the kth schema, X_k, has been generated.
   a. If $X_k.p_{k+1}$ is S-compatible, then let X_{k+1} be $X_k.p_{k+1}$
   b. If, in contrast, $X_k.p_{k+1}$ is not S-compatible, let X_{k+1} be $X_k. - p_{k+1}$.

D. Thus, each of the generated schemata, X_1, X_2, X_3, ... is a conjunctions of sentence letters and negations of sentence letters.

E. It is worth noting that, in contrast to the Rigid Plan, this procedure for generating conjunctions is not one that we can, in general, carry out in finitely many steps. This is because, since S has infinitely many elements, there may well be infinitely many truth tables that one would have to construct in order to show that any given schema is S-compatible.

VI. S-compatibility of the X_k's: basis step

A. Now we prove that all the X_k's are consistent with S. Since the X_k's are generated recursively, the proof is by induction.

B. The basis step is the claim that:

X_1 is S-compatible.

C. This is obvious if p_1 is S-compatible, since then X_1 is p_1.

D. But X_1 might also have been $-p_1$.

E. That would have been the case, however, only if p_1 is not S-compatible. This implies that there is a conjunction of members of S, call it W, such that $W.p_1$ is unsatisfiable.

F. Now, $-p_1$ either is S-compatible or is not. In the first case there's nothing we need to do.

G. Thus suppose that $-p_1$ is not S-compatible. Then, by definition there must be a conjunction Z of members of S such that $Z. - p_1$ is unsatisfiable.

H. Now comes the crucial move. We have now reached the conclusion that both $W.p_1$ and $Z. - p_1$ are unsatisfiable. By the general law of truth-functional logic mentioned above, $W.Z$ is then also unsatisfiable.

I. But, $W.Z$ is a conjunction of members of S. This contradicts the assumption we have made about S. Hence it cannot be the case that both p_1 and $-p_1$ are not S-compatible.

VII. S-compatibility of the X_k's: induction step

A. The induction step is really no more than a repetition of the basis step for the $k + 1$th schema generated.

B. The claim to be proven is that:

For any k, if X_k is S-compatible then so is X_{k+1}.

C. As before, the first question is, is $X_k.p_{k+1}$ is S-compatible? If so, then clearly $X_{k+1}{=}X_k.p_{k+1}$ is S-compatible.

D. Also as before, if $X_k.p_{k+1}$ is not S-compatible, but $X_k. - p_{k+1}$ is, then X_{k+1} is S-compatible.

E. So suppose that neither $X_k.p_{k+1}$ nor $X_k. - p_{k+1}$ is S-compatible. It follows that there are conjunctions W and Z of members of S such that $W.X_k.p_{k+1}$ and $Z.X_k. - p_{k+1}$ are both unsatisfiable. By the same general law cited above, $W.Z.X_k$ is unsatisfiable in which case X_k is not S-compatible, contradicting the induction hypothesis.

VIII. Compactness: verifying interpretation for S

A. As in the completeness proof, now we define an interpretation on the basis of the schemata generated.

B. We have just shown that X_k is S-compatible for each k. Let I be the truth functional interpretation that, for each k, assigns $\top$ to p_k if p_k is a conjunct of X_k, and $\bot$ to p_k if $-p_k$ is a conjunct of X_k.

C. Clearly, I, and **only** I, makes each X_k true. What we will now argue is that I makes every member of S true.

D. Let Z be any schema in S. The sentence letters of Z clearly are among the p_k's. Specifically, they are $p_{Z_1}, p_{Z_2}, \dots, p_{Z_k}$. The question is: is it possible that $I \not\models Z$?

E. Let $z = max\{Z_1, \dots, Z_k\}$, i.e., the highest serial number of the sentence letters that occur in Z.

F. As we have shown, I is the only interpretation such that $I \models X_z$.

G. If $I \not\models Z$, then there would be no truth value assignment that made both X_z and Z true.

H. But then, $Z.X_z$ is unsatisfiable, in which case X_z would not be S-compatible. But all X_k's are S-compatible; hence $I \models Z$.

I. Since Z is an arbitrary member of S, it follows that I makes every member of S true; hence, S is satisfiable. This conclude the compactness proof.