Philosophy 230

Wesleyan University
Fall 2014

Handout 11b

Universal Closure; Prenex Forms;
Completeness: The Central Lemma and the Rigid Plan

I. The Law of Universal Closure

A. The Law states that an open schema (i.e., with free variables) is valid iff its universal closure is, where the universal closure of an open schema is the result of universally quantifying all of the free variables in the open schema.

B. The argument for this law is straightforward. Let's look at an example. Consider $Rxy \vee -Rxy$. It should be obvious that this schema is $\top$ in all quantificational interpretations. What this means, however, is that for any arbitrary structure I, whatever objects, $a, b \in DQ$ are assigned to x and respectively to y, $I \vDash Rxy \vee -Rxy$.

C. Now, $J \vDash (\forall y)(Rxy \vee -Rxy)$ just in case for some $a \in DQ$ such that $x := a$, for every $b \in DQ$, if $y := b$ then $J \vDash Rxy \vee -Rxy$. By what we have just argued, this latter holds, hence for any interpretation J and $a \in DQ$ of J, if $x := a$ then $J \vDash (\forall y)(Rxy \vee -Rxy)$. That is, for all J, $J \vDash (\forall y)(Rxy \vee -Rxy)$.

D. The argument for the law simply generalizes the preceding reasoning to any schema Φ in which variables $x_1, \ldots, x_n$ occur free.

II. Prenex normal form

A. A schema is in prenex forms if all its quantifiers are in front:

$$(Q_1u_1)(Q_2u_2)\ldots\ (Q_nu_n)\ M$$

Here, each of the Q_i's is a quantifier, each of the u_i's is a variable, and M is a schema, containing as many variables as we like, which, however, contains no quantifiers.

B. So, for example, the schema

$$(\exists x)(\forall y)(\forall z)(Fx \supset Ryz)$$

is in prenex form. For it contains a string of quantifiers followed by a schema which contains no quantifiers. We call the string of quantifiers the *prefix*; the rest, the *matrix*.

III. Rules of Passage

A. These are laws for expanding the scope of the quantifiers, telling us how to bring a quantifier across some truth-functional operator. (That's why their discoverer, Jacques Herbrand, called them Rules of Passage.) There is a pair of rules, one for the universal and one for the existential quantifier, for each connective except the conditional, for which there are four rules.

B. The rules for negation are just the conversion of quantifier rules

C. The rules for conjunction are:

1. $p.(\exists x)Fx \Leftrightarrow (\exists x)(p.Fx)$
2. $p.(\forall x)Fx \Leftrightarrow (\forall x)(p.Fx)$

D. The rules for disjunction are:

1. $p \vee (\exists x)Fx \Leftrightarrow (\exists x)(p \vee Fx)$
2. $p \vee (\forall x)Fx \Leftrightarrow (\forall x)(p \vee Fx)$

E. The rules for the conditional are:

1. $p \supset (\exists x) \Leftrightarrow (\exists x)(p \supset Fx)$
2. $p \supset (\forall x)Fx \Leftrightarrow (\forall x)(p \supset Fx)$
3. $(\exists x)Fx \supset p \Leftrightarrow (\forall x)(Fx \supset p)$
4. $(\forall x)Fx \supset p \Leftrightarrow (\exists x)(Fx \supset p)$

IV. Prenexing

A. We can use these laws, together with the re-lettering law, to find a prenex equivalent of any schema. We shall show this by an example.

B. The procedure is simple: we move the quantifiers out, one-by-one, changing bound variables when we need to do so.

C. Example of Prenexing:

$$p \equiv (\forall x)Fx$$

D. We first transform it to:

$$(p \supset (\forall x)Fx).((\forall x)Fx \supset p)$$

E. Next, we move the quantifiers out from the two conjuncts:

$$(\forall x)(p \supset Fx).(\exists x)(Fx \supset p)$$

F. Note that we cannot bring either of these quantifier out without re-lettering, since then bound x would be within the scope of bound x. So we re-letter the first conjunct:

$$(\forall y)(p \supset Fy).(\exists x)(Fx \supset p)$$

G. And then take out both quantifiers:

$$(\forall y)[(p \supset Fy).(\exists x)(Fx \supset p)]$$

$$(\forall y)(\exists x)[(p \supset Fy).(Fx \supset p)]$$

Note here that the quantifier $(\exists x)$ goes *within* the scope of the quantifier $(\forall y)$, since it is the open sentence in the scope of that quantifier which is of the right form for the rule of passage.

H. Another example of prenexing:

$$(\forall x)(\exists y.Rxy \supset (\forall y)(Fy \vee Rxy))$$

V. The Completeness Theorem

> If A is a valid schema which contains no free variables, then A is deducible without any premises.

From now on we will call a schema in which no free variables occur a *closed* schema.

VI. The Central Lemma

> Suppose that R is a closed, prenex schema. If R is unsatisfiable, then we can deduce from it, by UI and strict EII, a finite number of quantifier-free schemata whose conjunction is truth-functionally unsatisfiable.

VII. Proof of Completeness from the Central Lemma

[1]	(1)	$-A$	P
		$\vdots$	
[1]	(k)	B	$\cdots$
		$\vdots$ [by Central	
		$\vdots$ Lemma]	
$[1, \cdots]$	(l_1)	Φ_1	$\cdots$
		$\vdots$	
$[1, e_1, \ldots, e_m]$	(l_n)	Φ_n	$\cdots$
$[1, e_1, \ldots, e_m]$	$(l_n + 1)$	$p. - p$	$(l_1) \ldots (l_n) TF$
$[1, e_1, \ldots, e_m - 1]$	$(l_n + 2)$	$p. - p$	$[e_m](l_n + 1) EIE$
		$\vdots$	
[1]	$(l_n + m + 1)$	$p. - p$	$[e_1](l_n + m) EIE$
[ ]	$(l_n + m + 2)$	$-A \supset p. - p$	$[1](l_n + m) D$
[ ]	$(l_n + m + 3)$	A	$(l_n + m + 2) TF$

$\Phi_1, \ldots, \Phi_n$ are quantifier free schemata. The schemata on lines $(e_1) \ldots (e_m)$ are introduced by applications of *strict EII*

VIII. Application of Central Lemma to obtain Completeness: example

Suppose we want to get a deduction of

$$(\forall x)(Fx \supset (\exists y)Fy)$$

[1]	(1)	$-(\forall x)(Fx \supset (\exists y)Fy)$	P
		$\vdots$	
[1]	(9)	$(\exists x)(\forall y)(Fx. - Fy)$	(8)EG; [3]EIE
[1, 10]	(10)	$(\forall y)(Fb. - Fy)$	(9)$bEII$
[1, 10]	(11)	$Fb. - Fb$	(10)UI
[1, 10]	(12)	$p. - p$	(11)TF
[1]	(13)	$p. - p$	[10](12)EIE
[ ]	(14)	$-(\forall x)[Fx \supset (\exists y)Fy] \supset p. - p$	[1](13)D
[ ]	(15)	$(\forall x)[Fx \supset (\exists y)Fy]$	(14)TF

IX. Equivalent of the Central Lemma for proof:

> Suppose that R is a closed prenex schema. Then if we can*not* obtain from R, by a number of applications of UI and strict EII, a finite number of quantifier-free schema whose conjunction is truth-functionally unsatisfiable, then R itself is satisfiable.

X. The Rigid Plan

A. Let a_1, a_2, and so forth be an infinite stock of variables which do not occur in the schema R at all.

B. For the first step of the procedure, we put down the schema R itself.

C. Then, at the next stage, we write down an instance of R, using the variable a_1.

1. First, for *every* variable *already introduced at any stage up through the one just completed*, we write down an instance, using that variable, of any universal schema generated, *again at any stage up through the one just completed* (unless we have already written them down).
2. Secondly, if at the *immediately preceding* stage, we wrote down any existential schemata, then we put down instances of these schemata, using the *next variables* in our list.

D. This completes our work at that stage, and we go on to the next, so long as two conditions are satisfied:

1. there is anything to do at the next stage, and,
2. the conjunction of the quantifier-free schemata so far produced is not unsatisfiable.

XI. Examples of the Rigid Plan

A. $(\forall x)(\forall y)(\exists z)(Fxz. - Fyz)$

B. $(\forall x)(\exists y)Fxy$