Philosophy 230

Wesleyan University
Fall 2014

Handout 10a

Soundness and Completeness;
First Part of Soundnes Proof

I. The Concepts of Soundness and Completeness

A. Are the validities and implications established by deduction the same as those defined in terms of interpretations?

1. Is it really the case that the schemata we deduce in the formal system, which depend upon no premises, are valid?
2. Is it really the case that if a schema is deduced from a set of other schemata, then the latter imply the former?
3. And, if so, how do we know that they are?

B. Using interpretations, we have defined a *semantic* notion of implication and validity; the notion is said to be semantic because it is connected with the meanings, or references, of expressions of different kinds.

1. On the other hand, using deductions, we have defined a *syntactic* notion of implication and validity; the notion is said to be syntactic because it is defined purely in terms of the *form* of a statement and pays no attention to what any of its parts might mean.
2. Our question, then, can be rephrased as follows: what is the relationship between the semantic notion of implication and the syntactic notion of implication? If a set of schemata implies another syntactically, do they imply it semantically? And what about *vice versa*?

C. Soundness:

If whenever a schema is deduced from a set of schemata, the former is semantically implied by the latter, then we say that the system of deduction is **SOUND**.

D. Completeness:

If whenever a schema is semantically implied by a set of schemata, the former can be deduced from the latter, then we say that the system of deduction is **COMPLETE**.

II. Proof of Soundness

A. For this proof we will show that if a schema A is deducible from a set of schemata $A_1, \ldots, A_n$, then it is implied by those schemata, in the sense of being $\top$ under every interpretation in which all those schemata are $\top$.

B. The proof has two parts.

1. First part: If a schema B is deducible from schemata $A_1, \ldots, A_n$, *without the use of EII and EIE*, then B is implied by $A_1, \ldots, A_n$.
2. Second part: If B is deducible from $A_1, \ldots, A_n$ ***with*** the help of EII and EIE, it is also deducible ***without*** the use of EII and EIE. That is, these rules of deduction are dispensable.

C. Notice that we will be going through a very slightly different proof from the text, because the rule we call Universal Generalization is actually the rule Goldfarb calls Liberalized Universal Universal Generalization. I will leave it as an exercise for you to show that our *UG* is eliminable in favor of Goldfarb's more limited *UG*. I will also not show that *EG* is dispensable.

III. Strategy of Part 1 of proof: induction

A. For the first part, we assume that we are given an arbitrary deduction Δ of a schema B, from schemata $A_1, \ldots, A_n$, in which all rules ***except*** *EII* and *EIE* are used.

B. Let us say that a line (k) is *good* if the schemata on the lines of the premises of line (k) imply the schema on line (k). We will try to show that every line of Δ is *good*.

C. Now, since Δ is from $A_1, \ldots, A_n$ to B, B occurs on the last line of the deduction, and the schemata occurring on the lines of its premise numbers are $A_1, \ldots, A_n$.

D. So, if every line of Δ is *good*, $A_1, \ldots, A_n$ imply B.

E. The strategy of the proof is mathematical induction:

1. We show first that line (1) is good. This is called the *basis step* of the induction.
2. Next, we assume that all the lines (1), (2), and so on, up to line $(k-1)$ are good. This assumption is called the *induction hypothesis*.
3. Then, we show that, given this assumption, if line (k) is deduced from lines (1)-$(k-1)$ by one of our rules, then line (k) is good, too. This is called the *induction step* of the proof.

F. It follows, then that the all lines of a deduction are good.

1. For line (1) is good.
2. Since line (1) is good, line (2) is good; since lines (1) and (2) are good, line (3) is good; and so forth.
3. Thus every line is good.

IV. Basis step: Line (1) is good.

A. What does line (1) look like? It can only be this:

$$[1] \quad (1) \quad A \quad P$$

B. So both the premise of (1) and the schema on (1) are A. But of course A implies A. So line (1) is good.

V. Cases of the Induction Step of Part 1 of the Proof of Soundness

A. For the induction step there are 8 cases, depending on which rule of deduction is applied to obtain line (k).

B. Case 1:
Line (k) is obtained by rule P:

$$[1] \quad (k) \quad A \quad P$$

C. Case 2:
Line (k) is obtained by rule D:

$$\begin{array}{llll} & & \vdots & \\ [i_1,\ \ldots,\ i_n, j] & (l) & S & \cdots \\ & & \vdots & \\ [i_1,\ \ldots,\ i_n] & (k) & R \supset S & [j](l)D \\ & & \vdots & \end{array}$$

D. Case 3:
Line (k) is obtained by rule TF:

$$[i_1,\ \ldots,\ i_n] \quad (k) \quad A \quad (j_1) \ldots\ (j_m)TF$$

E. Case 4:
Line (k) is obtained by rule EG:

$$\begin{array}{llll} & & \vdots & \\ [i_1,\ \ldots,\ i_n] & (l) & \Phi(y) & \cdots \\ & & \vdots & \\ [i_1,\ \ldots,\ i_n] & (k) & (\exists x)\Phi(x) & (l)EG \\ & & \vdots & \end{array}$$

F. Case 5:
Line (k) is obtained by rule UI:

$$\begin{array}{llll} & & \vdots & \\ [i_1,\ \ldots,\ i_n] & (l) & (\forall x)\Phi(x) & \cdots \\ & & \vdots & \\ [i_1,\ \ldots,\ i_n] & (k) & \Phi(y) & (l)UI \\ & & \vdots & \end{array}$$

G. Case 6:

Line (k) is obtained by rule CQ:

$$[i_1,\ \ldots,\ i_n] \quad (k) \quad A \quad (l)CQ$$

Here A may be one of four types of schemata, and the schema on line (l) will be of a corresponding type.

H. Case 7:

Line (k) is obtained by rule UG:

$$\vdots$$

$$[i_1,\ \ldots,\ i_n] \quad (l) \quad \Phi(y) \qquad \ldots$$

$$\vdots$$

$$[i_1,\ \ldots,\ i_n] \quad (k) \quad (\forall x)\Phi(x) \quad (l)UG$$

$$\vdots$$