How does Wittgenstein understand quantification in the *Tractatus*? In particular, what becomes of higher-order quantification? Higher-order quantification is central to Frege’s and Russell’s universalist conception of logic, the conception that Russell vividly encapsulates in *Introduction to Mathematical Philosophy*: ‘logic is concerned with the real world just as truly as zoology, though with its more abstract and general features’ (Russell 1919: 169). In particular, it is by means of quantification of predicate as well as singular term positions that the principles of logic abstract from the content that distinguishes the propositions of the special sciences. Logical principles get applied here in that their higher-order variables as well as their first-order variables may be instantiated by vocabulary from any science. This view of logic and its application implicates many features of the ‘old’ logic that Wittgenstein rejects in the *Tractatus*: a need for general principles to mediate specific inferences in argumentation (5.132), general validity as the mark of a logical law (6.1231), an important distinction between logical axioms and the truths derivable from them (6.127), to name a few.¹ The alternative the *Tractatus* offers to Frege’s and Russell’s universalist conception displaces higher-order quantification from the central role it had occupied. What happens to higher-order quantification itself in the *Tractatus* scheme?

Frege’s and Russell’s views of quantification are linked to their views of logical segmentation. On their views, there is no great divide as regards logical segmentation between atomic sentences and logically compound sentences. This comes out strikingly in Frege’s view that every sentence is multiply analyzable as the completion of an incomplete expression by type-appropriate completers, and in his assimilation in logic of the grammatical predicates of colloquial language to his first-level incomplete

¹ References to the *Tractatus* and *Prototractatus* are by Wittgenstein’s numbering system. In quotations I have variously used, combined, and emended the Ogden–Ramsey and Pears–McGuinness translations of the *Tractatus* and the McGuinness translation of the *Prototractatus*. 
expressions, formed by removing occurrences of a proper name from sentences of arbitrary complexity. Frege, without hesitation or argument, views his predicates as a kind of name, and in so doing, permits quantification of his predicate positions, treating quantification of predicate positions as fully parallel to quantification of proper name positions. The uniformity in Frege’s and Russell’s view of logical segmentation comes out as well in their view of the truth-functional connectives. For example, both Frege and Russell view material conditionals as relational sentences, so that the structure of a material conditional is parallel to that of ‘aRb’.

Wittgenstein thinks that this overarching uniformity in Frege’s and Russell’s view of logical segmentation leads them to overlook a distinction between two kinds of generality, generality proper and formal generality, as Wittgenstein calls them in *Prototractatus* 5.00533 and 5.005341. I will present an account of this distinction and say something about its significance for Wittgenstein’s philosophy of logic. The place to begin is with Wittgenstein’s view of the structure of expressions of truth-functionally compound sentences.

1 Two kinds of structure

Early on, Wittgenstein sharply distinguishes the structure of his elementary sentences from the iterative structure of truth-functional compounds of elementary sentences. Quantification, we shall see, is the product of interaction between these two different structures.

The structure of elementary sentences is the representing structure of a model. Wittgenstein epitomizes this view in the 1913 ‘Notes on Logic:’

In ‘aRb’ it is not the complex that symbolizes but the fact that the symbol ‘a’ stands in a certain relation to the symbol ‘b’. Thus facts are symbolized by facts, or more correctly: that a certain thing is the case in the symbol says that a certain thing is the case in the world. (NL 96[4]. See 4.0311, 3.21, 3.1432, and 2.15.)

The basic idea here is that in a sentence like ‘Socrates teaches Plato’, that ‘Socrates’ is related to ‘Plato’ in a particular way in the sentence—that ‘Socrates’ is the subject and ‘Plato’ the direct object of the verb ‘teach’ in the sentence—says that Socrates teaches Plato. It is the particular relation in which symbols stand in the sentence that expresses a sense, that represents a situation in logical space. In the material conditional

Socrates teaches Plato ⊃ Plato teaches Aristotle,

the antecedent and consequent express same sense as they do when they stand alone. Their role in the sentence is exhausted by each component sentence’s expressing the sense it does. Hence, in the conditional, the antecedent and consequent do not go proxy for anything. The relation in which the antecedent and consequent stand in the material conditional does not say anything. In this conditional, we encounter a kind of logical structure very different from the modeling structure of its antecedent and consequent. How does Wittgenstein understand this kind of structure?
An elementary sentence models a state of things (Sachverhalt). The sentence is true, if the state of things it models obtains; it is false, if the represented possibility does not obtain. Wittgenstein’s leading idea is that sentences generally are expressions of agreement and disagreement with the independent truth-possibilities of elementary sentences (4.4). We might form such an expression by taking several elementary sentence-models and surrounding them by indications of which of their truth-possibilities are the ones with which our new sentence agrees and which are the ones with which the new sentence disagrees. This may be done by the tabular notation Wittgenstein presents in the 4.3s and 4.4s. At 4.442 Wittgenstein presents a tabular expression for the material conditional ‘\( p \supset q \)’. The tabular notation removes the temptation to think of the arrangement of sentences around the hook as representing the holding of relation between items designated by ‘\( p \)’ and ‘\( q \)’ (4.441), and so makes it patent that the relationship of the elementary sentences indicated by ‘\( p \)’ and ‘\( q \)’ to this sentence is very different from the occurrence of a name in an elementary sentence.\(^2\)

We can extract from the tabular sign at 4.442 a particular way of forming a truth-function of two elementary sentences, of forming that truth-function which disagrees with the truth-possibility that the first elementary sentence is true and the second false, and agrees with the others. This way of forming a truth-function of two elementary sentences can be generalized to forming truth-functions of truth-functions of elementary sentences. From two truth-functions of elementary sentences \( \Phi \) and \( \Psi \), we can form that truth-function of elementary sentences that disagrees with any truth-possibility of elementary sentences that verifies \( \Phi \) and falsifies \( \Psi \), and agrees with all the others (5.31). In this way, we obtain intrinsically iterative truth-operations (5.3).

The sentences to which a truth-operation is applied to produce a truth-function of elementary sentences are the bases of the operation. 5.21 tells us that we can ‘... represent [darstellen] a sentence as the result of an operation that produces the sentence from other sentences (the bases of the operation)’ (cf. PTLP 5.001). This remark clues us into Wittgenstein’s understanding of the familiar Frege–Russell notation for truth-functionally compound sentences. Using the bracket notation of 5.501, let us consider how Wittgenstein would represent the construction of a truth-functionally compound sentence by means of successive applications of his N-operation, the truth-operation which yields a truth-function that agrees only with those truth-possibilities of elementary sentences which falsify each sentence among the bases of the operation.

Let \( \xi_1 = \text{elementary sentence } p \). Then \( \text{N}(\xi_1) = \sim p \).
Let \( \xi_2 \) have as values \( \text{N}(\xi_1) \) and elementary sentence \( q \). Then \( \text{N}(\xi_2) = p \land \sim q \).
Let \( \xi_3 = \text{N}(\xi_2) \). Then \( \text{N}(\xi_3) = \sim(p \land \sim q) = (p \supset q) \).

\(^2\) This point is emphasized in Sullivan (2000: 180–1).
This construction is summarized by the bracket expression:

\[ N(N(N(p), q)). \]

This bracket notation is, of course, a variant on Frege–Russell notation. Wittgenstein views a formula in Frege–Russell notation as representing the construction of a sentence from given elementary sentences by successive applications of truth-operations. Wittgenstein thus thinks of Frege–Russell formulas somewhat as we do, when we use them in metamathematical reasoning as shorthand for structural descriptions of object-language formulas. That is, Wittgenstein uses a description of the construction of a sentence via the iterated application of truth-operations to elementary sentences as an expression of the sense of that sentence. Descriptions of the construction of sentences are used for those very sentences.

This construal of Frege–Russell notation for truth-functions yields a consequence that will prove central to Wittgenstein’s account of quantification. A sentence is an expression of agreement and disagreement with the truth-possibilities of elementary sentences (4.4), and so an expression of a particular truth-function of given elementary sentences. Sentences that express the same truth-function of the same elementary sentences have the same sense. 3.34 distinguishes between ‘the accidental features of sentences that arise from the particular way the sentence-sign is produced’ and the ‘essential features which are required in order to enable the sentence to express its sense’. As Wittgenstein understands them

\[ p \supset q, \]
\[ \sim p \lor q, \]

and

\[ N(N(N(p), q)) \]

all portray different successive applications of truth-operations to ‘p’ and ‘q’ that yield the same truth-function of ‘p’ and ‘q’. When we use these formulas as expressions for this truth-function, we find ourselves with three sentences expressing the same sense.\(^3\) The features of these sentences that characterize a particular construction of the truth-function they all express are accidental features of the sentences. In particular, the occurrence of a particular molecular subformula in a Frege–Russell sentence like the examples above is generally an accidental feature of the sentence, for the occurrence of a molecular subformula typically characterizes only an initial segment of a particular construction of a truth-function, not the truth-function constructed.

\(^3\) 5.141 says, in effect, that sentences expressing the same sense are the same sentence. We might then say that we have here three different sentence-signs belonging to the same sentence, to the same symbol.
2 Sentence-functions and quantification

Wittgenstein says very little about quantification in *Tractatus*, and nothing explicitly about higher-order quantification. What little he says explicitly about quantification is concentrated in the 5.52s. There Wittgenstein tells us:

I separate the concept all from truth-functions.

Frege and Russell introduced generality [Allgemeinheit] in connection with logical product or logical sum. This made it difficult to understand the sentences ‘(∃x).fx’ and ‘(∀x).fx’, in which both ideas are embedded.

By *Tractatus* standards, this remark is fairly straightforward. As Wittgenstein views matters, a Frege–Russell generalization expresses the result of applying a truth-operation to its instances. There are two facets to this expression: the collection together of the instances of the generalization—the concept all—and the indication of a particular truth-operation.

5.501 gives us Wittgenstein’s view of the first task:

When a bracket-expression [Klammerausdruck] has sentences as its terms—and the order of the terms inside the brackets is indifferent—then I indicate it by a sign of the form ‘(x)’. ‘x’ is a variable whose values are terms of the bracket-expression and the bar over the variable indicates that in the brackets it goes proxy for (vertreten) all its values.

(Thus, if x has three values P, Q, and R, then

\[(\bar{x}) = (P, Q, R)\].

The values of the variable is something to be stipulated.

The stipulation is a description of the sentences that the variable goes proxy for.

How the description of the terms of the bracket-expression is produced is not essential.

We can distinguish three kinds of description: 1. direct enumeration, in which case we can, instead of the variable, simply put in its constant values; 2. giving a function fx whose values for all values of x are the sentences being described; 3. giving a formal law that governs the construction of the sentences, in which case the bracket-expression has as its terms all the terms of a series of forms.

My earlier discussion of Wittgenstein’s understanding of Frege–Russell notation for truth-functions illustrates the first way of stipulating the values of a variable. Quantification involves the second—the description of the bases of a truth-operation by means of a function, a propositional function or, as I shall say, sentence-function.4

The *Tractatus* offers no account of sentence-functions. I take the context for the notion to be provided in the 3.3s by Wittgenstein’s notion of an expression. 3.3 is Wittgenstein’s version of the context principle: ‘Only sentences have sense; only in the nexus of a sentence does a name have meaning.’

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4 Two terminological points. First, to avoid confusion with Russell’s pre-1910 notion of a proposition as well as with contemporary conceptions, I translate ‘Satz’ by ‘sentence’. Second, as Peter Hylton (1997) has observed, Wittgenstein follows Whitehead and Russell’s usage, and uses ‘function’ in the *Tractatus* to mean ‘propositional function’. Like Russell, these are the only functions Wittgenstein recognizes.
3.31 I call any part of a sentence that characterizes its sense an expression (or a symbol).
   (A sentence is itself a symbol.)
   Everything essential to their sense that sentences can have in common with one another is an expression.
   An expression is the mark of a form and a content.
3.318 Like Frege and Russell, I conceive of a sentence as a function of the expressions contained in it.

4.24 indicates the application of these ideas to elementary sentences:
I write elementary sentences as functions of names so that they have the form ‘fx’, ‘φ(x, y)’, etc.

So, let us begin with elementary sentence-functions.
To fix ideas, I will assume that elementary sentences have Russellian structures, asserting that individuals stand in assorted n-adic relations.5 Something elementary sentences can have in common that characterizes their sense is the occurrence of a name. Names must then be expressions. Names are not the only expressions in elementary sentences. Two elementary sentences might be just alike except that one contains occurrences of one name where the second contains occurrences of another name:

\[ aRb; \]
\[ cRb. \]

Here the common expression properly contains a name. I’ll call such expressions of elementary sentences ‘predicates’. Wittgenstein’s understanding of the expressions contained in elementary sentences is, then, in its way Fregean.6 We can view the elementary sentence as a function of a name, holding the predicate fixed. We can also view it as a function of a predicate, holding a name fixed. Any elementary sentence is a value of various sentence-functions. These possibilities of analysis are intrinsic to the structure by virtue of which elementary sentences present states of things. It is this structure that makes elementary sentences functions of the expressions they contain.

We can then form a sentence-function from an elementary sentence by converting any name or predicate in it into a variable-expression. We can use these elementary sentence-functions to stipulate values of sentence-variables to serve as bases for an application of a truth-operation that yields a truth-function of those values.

For example, let \( \xi_1 \) have as values the values of the sentence-function ‘x teaches Plato’. Then \( N(\xi_1) = \neg(\exists x)(x \text{ teaches Plato}) \). Let \( \xi_2 \) have as its value \( N(\xi_1) \). Then \( N(\xi_2) = \)

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5 There has been a long-running debate whether, on the assumption that elementary sentences have Russellian structures, in e.g. ‘Socrates teaches Plato,’ the relation between ‘Socrates’ and ‘Plato’ is itself a name in the sentence. My purposes here do not require me to take a stand on this issue. Following Wittgenstein’s usage in 4.24, I will restrict myself to examples involving names of individuals.

6 See especially Frege (1879: §9), where Frege says that an expression of a judgment can be regarded as a function of the signs that occur in it. This is the only place in his writings where Frege applies the word ‘function’ to linguistic expressions.
(∃x)(x teaches Plato). As before, we might collapse the series of stipulations of values for variables into an embedded bracketed expression prefixed by an indication of a truth-operation:

\[ N(N(x \text{ teaches Plato})). \]

By means of these simple examples, Wittgenstein indicates his analysis of Frege’s and Russell’s quantifiers, at least their first-order quantifiers (5.52). These simple examples, however, lead directly to the issue of the Fogelin–Geach exchange, namely Wittgenstein’s understanding of multiply embedded quantifiers. Here I side with Geach and those early readers of *Tractatus*—Russell, Ramsey, and Carnap—none of whom saw any difficulty here.

To begin, how might the truth–function expressed in Frege–Russell notation by ‘(∀x)(x teaches Plato)’ be portrayed using sentence–functions and the N–operation? We could obtain this truth–function by applying the N–operation to the totality consisting of the results of applying the N–operation individually to each value of the sentence–function ‘x teaches Plato’. But how, Fogelin asks, are we to stipulate the values for a variable whose values are these individual negations? 5.32 states:

All truth–functions are results of successive applications to elementary sentences of a finite number of truth–operations.7

Fogelin urges (1987: 80) that this thesis rules out the unordered applications of a truth–operation to potentially infinitely many sentences. This, however, is exactly what the description of the desired bases appears to require.

I maintain that Wittgenstein approaches this issue from a Fregean direction. To obtain the desired bases, we first apply the N–operation to a representative arbitrary value of our sentence–function, say ‘Socrates teaches Plato’, to get the truth–function

\[ N(\text{Socrates teaches Plato}). \]

Conversion of the name ‘Socrates’ into a variable–name yields a sentence–function whose values are the results of applying the N–operation to each value of ‘x teaches Plato’. A further application of the N–operation to the bases given by this sentence–function yields the truth–function expressed by our universal quantification. This Fregean approach both avoids the unordered applications of a truth–operation to potentially infinitely many sentences, and is continuous with the understanding of elementary sentence–functions just sketched. The form common to an elementary sentence and the state of things it represents makes it possible to remove a name from an elementary sentence and so to transform a picture into a proto–picture—an Urbild, a pattern or form—that collects together all those elementary sentences that flesh out this pattern. Suppose we replace a name in an elementary sentence that occurs within the representation of the construction of a particular truth–function of that elementary

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7 The idea is repeated at 6.001, a comment of the general form of sentences exhibited in 6.
sentence. The elementary sentence-function in the formula collects together its values so that the entire formula collects together expressions of that particular truth-function of values of the elementary sentence-function. In this way then, we have not only elementary-sentence functions, but also sentence-functions formed by conversion of occurrences of expressions in elementary sentences that occur in portrayals of truth-functions into variable expressions. Although 4.0411 demonstrates that Wittgenstein knew how the features of quantifier-variable notation suit it to the expression of multiple generality, he does not himself in the *Tractatus* introduce notation adequate to the use of these sentence-functions to stipulate values of sentence-variables. As Peter Geach observed, it is easy to supplement Wittgenstein’s notational suggestions and usage in the 5.5s to provide such a notation. I have noted how we can portray truth-functions of elementary sentences by embedded bracket-expressions. We need a device in these embedded bracket-expressions to indicate the stage at which a name in elementary sentences is converted into a variable-name to yield a sentence-function. To this end, in a bracket-expression I will write the variable-expression in boldface outside the nexus of an elementary sentence next to the representation of the sentence-function obtained by use of the variable-expression. So,

\[
N(x \ x \text{teaches Plato}) = \neg (\exists x) (x \text{teaches Plato})
\]

\[
N(N(x \ x \text{teaches Plato})) = (\exists x) (x \text{teaches Plato})
\]

and

\[
N(x \ N(x \text{teaches Plato})) = (\forall x) (x \text{teaches Plato})
\]

With these sentence-functions, Wittgenstein can capture multiply embedded quantification over objects. For instance, ‘(\forall y)(\exists x)(x \text{teaches } y)’ goes over into

\[
N(y \ N(x \ x \text{teaches } y))
\]

This embedded bracket-expression portrayal encapsulates the following construction. We begin with an arbitrary instance of the elementary sentence-function ‘x teaches y’, say, ‘Socrates teaches Plato’. Let \(\xi_1\) have as its values the values of the sentence-function ‘x teaches Plato’. So \(N(\xi_1) = N(x \ x \text{teaches Plato})\). Let \(\xi_2\) have as its values the values of the sentence-function ‘N(x x teaches y)’. The desired sentence is \(N(\xi_2) = N(y \ N(x \ x \text{teaches } y))\). This embedded bracket-expression notation is thus a variant on Frege’s and Russell’s first-order quantifier-variable notation. It shows us how to understand Frege–Russell quantifier–variable formulas as representations of truth-functions of elementary sentences.

There is a complication here that deserves mention. In the 5.53s, Wittgenstein rejects use of a sign for identity (as a primitive sign) in favor of expressing the identity of objects by means of the identity of names for those objects, difference of object by difference of

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8 See Geach (1981). Geach, however, brings in the notion of a class of sentences to construe his version of Klammerausdruck notation. In contrast, I believe that talk of classes of sentences is to be understood in terms of the stipulation of sentence-variables. The notion of a sentence-variable is prior to that of a class of sentences.
names. Identity and difference of variable-names is similarly treated. Multiply quantified sentences thus get a non-standard construal. For example, as Wittgenstein understands it, the above portrayal of \( (\forall y)(\exists x)(x \text{ teaches } y) \) says that everyone is taught by someone else. I won’t be concerned here about the rules for interpreting bracket expressions in conformity with Wittgenstein’s views on identity. Perhaps the complications that his views on identity introduce into the construal of multiply embedded quantifiers explain why Wittgenstein’s examples in the 5.52s are all monadic quantifications.\\footnote{For a thorough and instructive discussion of the Tractatus treatment of quantification and identity, see Brian Rogers and Kai F. Wehmeier (forthcoming). I agree with Rogers and Wehmeier that their weakly exclusive interpretation is the view best attributed to the Tractatus.}

Wittgenstein’s analysis of first-order quantification exploits the use of sentence-functions obtained from portrayals of truth-functions to stipulate the bases for further applications of truth-functions. We see here how the interactions between the picturing structure of elementary sentences and the iterative structure of portrayals of truth-functions add to the expressive power of language. As long as we are stipulating values of sentence-variables either by lists of sentences or lists of elementary sentence-functions in the same variables, the tabular notation of the 4s suffices, and the iterative possibilities of truth-operations are idle. The formation of sentence-functions from truth-functions and their use to describe the bases for further truth-operations opens the way to expressing truth-functions that are not expressible in the tabular notation. Of course, whether there are truth-functions not expressible in tabular notation depends on the forms of elementary sentences and the number of names.\\footnote{In my discussion of embedded first-order quantification in Tractatus, I have benefited from the discussions in Kremer (1992) and Floyd (2001).}

3 Form-series variables and higher-order quantification

I have been explaining Wittgenstein’s analysis of first-order quantification, quantification of name positions, in terms of sentence-functions. We have seen how Tractarian sentence-functions gather together the instances of first-order generalizations to serve as bases of truth-operations. I have restricted my examples to sentence-functions whose arguments are names, indeed names of individuals, given my assumptions concerning the forms of elementary sentences. Everything I’ve said about the analysis of quantification of name-positions holds equally for quantification of predicate-positions in elementary sentences. The arguments for these sentence-functions are predicates from elementary sentences.\\footnote{In particular, the predicate-quantifier in 5.5261 can naturally be taken to be a quantification of a predicate-position in an elementary sentence.}

We will not then be able to capture Frege’s or Russell’s second-order quantification using these sentence-functions, because the substituends for these variables are predicates of arbitrary truth-functional and first-order complexity.

Wittgenstein conceives of generalizations as truth-functions of their instances. This conception rules out Frege’s impredicative higher-order quantification on pain of
vicious circularity. What about Russell’s predicative higher-order quantification? At this stage, it looks tempting to accommodate higher-order quantification by following Russell’s example and admitting sentence-functions that take non-elementary sentence-functions as arguments. We then could form a sentence-function

$$(\forall x)(x \text{ teaches Plato } \supset \Psi(x)),$$

whose values are the results of replacing ‘$\Psi(x)$’ by any first-order formula containing only ‘$x$’ free. So the values for this putative sentence-function would include:

$$(\forall x)(x \text{ teaches Plato } \supset \neg(x \text{ is mortal}))$$
$$(\forall x)(x \text{ teaches Plato } \supset (\exists y)(x \text{ teaches } y \land y \text{ teaches } x))$$
$$(\forall x)(x \text{ teaches Plato } \supset (\forall y)(x \text{ loves } y \supset (\exists z)(y \text{ loves } z))).$$

I claim that Wittgenstein’s view of expressions and sentence-functions does not permit sentence-functions that gather together the instances of Russell’s predicative second-order generalizations. The arguments for sentence-functions are expressions, and expressions characterize the senses of the corresponding values of those functions. So far we have recognized as expressions the names and predicates of elementary sentences. It should be clear from the discussion in §2 how these expressions characterize both the senses of those elementary sentences as well as the senses of truth-functions of those elementary sentences. So, do the substituends for ‘$\Psi(x)$’ generally characterize the corresponding instances of our sample schema

$$(\forall x)(x \text{ teaches Plato } \supset \Psi(x))?$$

On the Tractarian construal of Frege–Russell first-order formulas, a molecular substituend for ‘$\Psi(x)$’ indicates only an initial segment in the construction of the sentence which is portrayed by the entire formula. More precisely, each such substituend is the result of replacing a name with a variable-name in the portrayal of the construction of a sentence, a sentence which in turn is the basis for the application of a truth-operation in the construction of the corresponding instance of our schema. In general, an initial segment in a portrayal of a construction of a sentence does not characterize the sense of that sentence: it characterizes neither the elementary sentences of which that sentence is a truth-function, nor the specific truth-function of those elementary sentences that the sentence expresses. Wittgenstein’s discussion of the construction of sentences indicates as much. 5.25 tells us the occurrence of an operation in the construction of a sentence does not characterize its sense. 5.501 states that in portraying the application of a truth-operation to bases, how the bases are described is inessential (unwesentlich). Inessential for what? It is inessential as regards (the sense of)

12 An impredicative predicate-variable over Fregean concepts may be instantiated by a predicate containing a quantified variable over those same concepts.

13 In some cases, a subformula does characterize the sense expressed by a sentential-sign in which it occurs, for example, if the subformula is equivalent to an elementary sentence or if the entire sentence is equivalent to the subformula.
the sentence produced by the application of the truth-operation to those bases. All this is an elaboration of the discussion of Wittgenstein’s construal of Frege–Russell notation for truth–functions at the end of §1.

There is one question that remains. I have said that the values of the sentence-function

\[ \varphi(a) \]

are all the elementary sentences containing the name ‘\( a \)’. But according to 3.313, the sentence–function representing an expression has as values all the sentences containing the expression. Truth–functions whose truth–arguments include elementary sentences containing the name ‘\( a \)’ themselves contain that name. Yet on my account these are not in general among the values of

\[ \varphi(a).^{14} \]

4.23 observes that names—and so elementary sentence predicates—occur in sentences only in the context of elementary sentences. I earlier argued that it is the occurrence of expressions of elementary sentences in sentences generally that yield a sentence–function like

\[ \neg(x \text{ teaches Plato}), \]

whose values are negations of the instances of the elementary sentence–function

\[ x \text{ teaches Plato.} \]

5.442 states:

If we are given a sentence, then with it we are also given the results of all truth–operations that have it as their base.

5.47 elaborates:

An elementary sentence really contains all logical operations in itself. For ‘\( fa \)’ says the same thing as ‘\( (\exists x).fx.x=a' \).

Where there is compositeness [Zusammengesetztheit], argument and function are present, and where these are present, we already have all the logical constants.

I suggest that Wittgenstein views the sentence–function whose values are the elementary sentences containing ‘\( a \)’, to encompass virtually all sentences containing ‘\( a \)’. At the end of the chapter, I will return to say something about how ‘virtually’ should be understood here.

Let us return to the topic of higher–order predicative quantification. I have argued that Tractarian sentence–functions cannot be used to capture Russell’s predicative second–order quantifications. Still, we can recognize in formal terms that exploit the

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14 I am grateful to Michael Potter for raising this issue.
iterative possibilities of truth-operations the Frege–Russell sentences that are instances of predicative second-order generalizations. It seems that we should be able to stipulate these sentences to be values of a sentence-variable and so to serve as the bases for a truth-operation. And so we can. The instances of a Russellian second-order generalization are not the values of a sentence-function. Rather, they are the members of a form-series (Formenreihe). Wittgenstein’s third way from 5.501 for describing the values of variables gives him the resources to simulate higher-order quantification.

5.501 says that we can describe the sentences that are to be the values of a variable, the terms of a bracket-expression, by ‘giving a formal law that governs the construction of the sentences, in which case the bracket-expression has as its terms all the terms of a series of forms’. I shall call variables whose values are stipulated by formal laws, form-series variables. Wittgenstein says very little about this third means for stipulating the values of a variable. It is linked to his notion of an operation. An operation is a logically significant recursive notational procedure for generating sentences from sentences (see the 5.23s). At 4.1273, we find an informal presentation of Wittgenstein’s only example of a form-series variable:

If we want to express in begriffsschrift the general sentence, ‘b is a successor of a’, then we require an expression for the general term of the form-series:

\[ aRb, (\exists x): aRx . xRb, (\exists x,y): aRx . xRy . yRb, \ldots \]

The general term of a form-series can only be expressed by means of a variable, for the concept: member of this form-series, is a formal concept. (Frege and Russell overlooked this. The way they want to express general sentences like the above is therefore incorrect; it contains a vicious circle.)

We can determine the general series of forms by giving its first term and the general form of the operation that produces the next term out of the sentence that precedes it.\(^{15}\)

I’m not going to be concerned here with the formal details of specifying form-series. Göran Sundholm, after raising assorted difficulties in interpreting Wittgenstein’s theses, examples, and notational scraps on this topic, concludes:

The author of the *Tractatus* . . . constitutes the finest example of a philosopher whose technical formal capacities do not reach the outstanding level of his logico-philosophical thinking. (Sundholm 1992: 76)

I will assume that wherever there is a procedure for recognizing members of a class of Russellian first-order sentences in terms of logically significant features of their construction, there is a formal law that generates a form-series whose members are these sentences.\(^{16}\)

\(^{15}\) In *Prototractatus*, a version of this remark (PTLP 5.005342, 5.00535, and 5.005351) follows very closely the introduction of formal generality in 5.00531.

\(^{16}\) Two points. First, specification of the form-series may, as in 4.1273, require imposing a logically arbitrary alphabetic ordering on signs. This is permitted so long as any alphabetic ordering of the same signs could be employed to specify a form-series of the same terms. Second, my talk of a procedure to recognize the members of a class of sentences is ambiguous between decision procedures (recursive) and search procedures (semi-recursive). For my purposes here, all I need is the weaker reading. Nothing I say excludes the stronger reading.
Suppose we have, then, a form-series whose members are the first-order sentences formed from some countable stock of names. There will then be a form-series of sentence-functions of a single variable individual-name formed from these sentences. These terms are the substituends for Russell’s second-order variables ranging over, as Russell would put it, first-order propositional functions of a single individual. We can now exploit this form-series to construct that series of sentences we get when we replace \( \forall x (x \text{ teaches } \text{Plato} \supset \Psi(x)) \),

first by the first term in our enumeration of sentence-functions, then by the second term, then by the third term. That is, we have a form-series constructed from the form-series enumerating first-order sentence-functions which enumerates the Russellian instances of \( \forall x (x \text{ teaches } \text{Plato} \supset \Psi(x)) \). We can stipulate a variable with the members of the form-series as values. Application of a truth-operation to the bases given by this variable thus simulates Russellian second-order quantification.  

In general, to represent the construction of the application of a truth-operation to the terms of a form-series, first construct the first term of the form-series. At the next stage, use a form-series variable to gather the members of the form-series and apply the desired truth-operation to it. Understood in this way, the simulation of embedded Russellian second-order quantifications poses no problems. Consider a second-order formula of the form

\[
(\forall \Phi)(F[\Psi] \supset (\exists \Phi)G[\Psi, \Phi]),
\]

where ‘\( F[\ ] \)’ and ‘\( G[\ , \ ] \)’ are first-order matrices. Let us suppose that ‘\( x \text{ is mortal} \)’ is the first sentence-function in our form-series of first-order sentence-functions. So, first, we construct the first-order sentence

\[ G[x \text{ is mortal}, x \text{ is mortal}] \]

Now introduce a form-series variable whose first term is our sentence and whose \( n \)th term is the result of substituting for the second occurrence of ‘\( x \text{ is mortal} \)’ the \( n \)th sentence-function in our enumeration, and apply generalized disjunction to the bases given by this variable. This gives us the truth-function portrayed by:

\[ (\exists \Phi)G[x \text{ is mortal}, \Phi] \]

This sentence can now be a basis for further truth-operations. Moreover, sentence-functions can be obtained from it via the replacement of expressions within elementary sentences by variable-expressions. We can then go on to construct the sentence:

\[ F[x \text{ is mortal}] \supset (\exists \Phi)G[x \text{ is mortal}, \Phi]. \]

Consider the form-series whose \( n \)th term is the result of substituting the \( n \)th sentence-function in our enumeration in this frame for ‘\( x \text{ is mortal} \)’. An application of generalized conjunction to the terms of this series gives us the desired truth-function. Our

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17 In this use of form-series to simulate higher-order quantification, I benefited from conversations with Michael Potter and Peter Sullivan.
truth-function is then a generalized conjunction, whose conjuncts are conditionals. The antecedent of each conjunct is a first-order sentence. The consequent is a generalized disjunction, each disjunct of which is a first-order sentence.

Our form-series enumerating first-order sentence-functions thus gives us simulations of Russellian second-order quantifications over first-order propositional functions of a single individual. We must, however, remember that the second-order variables are not Tractarian variable-expressions used in the portrayal of sentence-functions in stipulations of the values of a sentence-variable. They are, as I shall say, pseudo-variables, to be understood in terms of the form-series determined by the form-series enumerating first-order sentence-functions and the construction of the formula, the bracket expression, that fills the scope of the pseudo-variable. We may, nevertheless, incorporate these pseudo-variables into the bracket notation for portraying truth-functions of elementary sentences, if we wish. We can now formulate a formal law that generates all sentence-functions that take a single individual name as argument in this extended notation. These expressions are the substituends for Russell’s third-order variables generalizing over, as he puts it, second-order propositional functions of a single individual. We can then repeat the preceding construction to simulate Russellian fourth-order quantifications by introducing a new type of pseudo-variable. And so on. In this way, modulo the original form-series generating a class of first-order sentences, we can simulate full predicative second-order logic. I see no bar to the construction of the entire ramified hierarchy in Tractarian terms. I would like to think that this procedure is what Wittgenstein has in mind, when he speaks in 5.252 of advancing from type to type in the hierarchies of Russell and Whitehead.

I spoke of the *Tractatus analysis* of first-order quantification, but only of its *simulation* of Russellian second-order quantification. I approached Wittgenstein’s treatment of quantification by searching for Tractarian resources for stipulating variables whose values are the instances of the desired quantifications. In the case of first-order quantifications, sentence-functions gathered together the desired instances. Sentence-functions are not, however, guaranteed to gather together sentences that are the result of arbitrarily many applications of truth-operations to elementary sentences. So, to accommodate higher-order quantification, I turned to form-series. A Russellian variable over second-order propositional functions of a single individual may be instantiated by any first-order monadic predicate of arbitrary logical complexity. My Tractarian simulation of second-order quantification restricts the substituends for my second-order pseudo-variables to the sentence-functions present in a given form-series. There is no guarantee that there is a form-series whose members include all first-order sentences. That will depend on the forms of elementary sentences and the number of names. The generality present in my Tractarian simulation of higher-order

\[\text{Here in addition to pseudo-variable sentence-functions, we will need to introduce notation for pseudo-higher-order sentence-functions that take pseudo-lower-order sentence-functions as pseudo-arguments.}\]
quantification is formal generality, not quantificational generality with its source in the representing structure of elementary sentences.

This difference between proper quantificational generality and formal generality has a further consequence. For Russell, all generality is captured by quantification of positions defined within the ramified theory of types. While the hierarchy of the ramified theory can be simulated by the use of form-series, form-series can also be used to express truth-functions that are not expressible with the ramified theory. Indeed, the use of form-series to define the ancestral of a relation over countably many objects is an example. The same sort of construction used to simulate Russellian second-order quantification can be applied to any form-series whose members are truth-functions of elementary sentences obtained by arbitrarily many applications of truth-operations and so arbitrarily many embeddings of scopes of variable-expressions in elementary sentences. Nothing logically distinguishes the simulation of higher-order quantification from other uses of form-series.

4 The general form of sentences

The account I have given of the sources of generality in the representing structure of elementary sentences and the iterative possibilities of truth-operations sheds light on the proper understanding of the general form of sentences. The general form of sentences is presented at 6:

The general form of a truth-function is: \[ \bar{p}, \xi, N(\xi) \].

This is the general form of sentences.

This specification of the general sentence-form presents several problems. The one most salient in the present context is that the general form of sentences says nothing about how the bases for later applications of truth-operations depend on previously constructed sentences.

I have given several examples of constructions of sentences, constructions of truth-functions of elementary sentences from elementary sentences. Each construction begins with elementary sentences, given individually or by elementary sentence-functions. Each step in the construction has two stages: stipulation of a sentence-variable and the application of a truth-operation to the terms the variable goes proxy for. For the construction to be a stepwise construction, we must at each stage stipulate the values of that stage’s sentence-variable in terms of what has gone before. This does not mean that the sentences represented by the variable stipulated at stage \( n \) must have been constructed at an earlier stage. This will be the case only if the variable is stipulated by means of a list. We may stipulate a variable at stage \( n \) by means of a sentence-function obtained from a sentence constructed at an earlier stage. To have constructed a sentence is to have available the sentence-functions obtainable from it. After all, the instances of the sentence-function are constructible in the same way as the value of it that appears at an earlier stage in the construction. Here then we have a potentially
infinitary step in the construction of sentences. Wittgenstein treats form-series stipulations in a parallel way. Having constructed a sentence at an early stage, we may at a later stage stipulate a sentence-variable by means of a formal law for a form-series whose first member is that sentence. The application of a truth-operation to the values of a form-series variable counts as a single step, even though arbitrarily many steps may be required for the construction of the individual terms of the form-series. Form-series then introduce another potentially infinitary dimension into sentence construction.

These examples suggest that the general form of sentences has an intrinsically schematic character.\textsuperscript{19} Wittgenstein does say that the general sentence-form is a variable (4.53), and uses the notation for form-series introduced at 5.2522 to specify it at 6. Nevertheless, I do not take this specification itself to be the specification of sentences via a form-series, at least not in the way that my earlier examples of the use of form-series variables in the construction of sentences are. The general sentence-form is rather a schema for the construction of any sentence: it is the most general form for the construction of truth-functions of elementary sentences. I take 6.001 to indicate as much in commenting on the notation for the general sentence-form that 6 presents:

This says nothing other than that every sentence is the result of successive applications of the operation $N(\xi)$ to elementary sentences.

So, the opening term of the general form is an indication that the construction of sentences begins with some elementary sentences. The ways in which elementary sentences may be specified, apart from lists of individual sentences, depends on their forms, the forms of their constituting names, and the number of names of each form. Hence, the general form is properly silent here (see 5.55–5.551). The general form is also silent as to how a later specification of the bases for an application of the N-operation must depend on sentences constructed earlier. The bases are specified by stipulating the values of a sentence-variable. The only requirement here is that the stipulation of the values of the variable ‘is a description of the symbols and states nothing about what is signified’ (3.317).\textsuperscript{20}

3.317 requires that the description be in terms of symbols and so pertain to the senses expressed by the values of the stipulated variable.\textsuperscript{21} It is clear how the stipulation of values for a variable by means of a sentence-function satisfies this condition: not only the arguments for a sentence-function but the sentence-function itself characterizes the senses of the values of that sentence-function. What about a stipulation by means of a form-series? The sentences of a form-series are the sentences produced by successive applications of an operation (5.2521–5.2522). An operation, 5.241 says, does not characterize a form, but a

\textsuperscript{19} I’m indebted here to Juliet Floyd, who years ago in an unpublished paper urged the schematic character of the general sentence-form.

\textsuperscript{20} The antecedent of this passage in PTLP is 5.0052, and is a part of the material from 5.003–5.005341 that contains the antecedents of TLP 5.501.

\textsuperscript{21} This requirement is consistent with my claim in §3 that the way in which the bases for an application of a truth-operation are described does not characterize the sense of the sentence thereby produced.
difference of forms: it is a formal relation between the bases of the operation and its result (5.22 and 4.122). I suggest that two sentences share the same logical form, if both are the same truth-function of elementary sentences of the same form. The logical form of an elementary sentence corresponds to the sentence-function obtained by replacing each name in the elementary sentence with a variable-name (see 3.315). Consider the N-operation. It yields a sentence which is true just in case the bases to which it is applied are all false. This is the difference it marks—or the relation it expresses—between its bases and its results. Consider now the operation that generates the bases for the truth-function which says that some object $b$ $R$-follows $a$. This operation, applied to the conjunction of disjunctions which says, so to speak, that $n$ distinct steps along the relation $R$ suffice to get from $a$ to $b$, yields the conjunction of disjunctions which says that it takes $n + 1$ steps. The difference between $n$ steps and $n + 1$ steps along a relation is the difference in form, the difference in sense, marked by this operation. Differences in logical form are differences in sense. Since operations mark a difference in form as between bases and result, the specification of a form-series in terms of the iterated application of an operation does involve a description of symbols, and not just signs.

I can now redeem an earlier promise. I noted how the values of the elementary sentence function

$$\varphi(a)$$

are only the elementary sentences containing ‘$a$’, and not the truth-functions of those elementary sentences, whereas, according to 3.313, the sentence-function representing an expression has as values all the sentences containing the expression. I suggested that the elementary sentence-function virtually encompasses all the sentences containing ‘$a$’. I can now explain what ‘virtually’ comes to here. The variable whose values are all the sentences containing ‘$a$’ will be the restriction of the general form of sentences to bases which include some elementary sentences that are values of the elementary sentence-function ‘$\varphi(a)$’.\(^{22}\)

If my suggestion about the schematic character of the general sentence-form is correct, the notation of 6 is not an example—or not merely an example—of Wittgenstein’s limitations as a designer of notation. Rather, in light of the exposition of the 5s on the theme, ‘A sentence is a truth-function of elementary sentences,’ 6 crystallizes the open-ended presentation of the general form from 4.5: ‘Es verhält sich so und so.’ That’s the way it is.\(^{23}\)

\(^{22}\) I take Anscombe (1959: 97) to voice a similar idea.

\(^{23}\) I have benefited from conversations on the topics of this chapter with Enzo DePellegrin, Juliet Floyd, Peter Hylton, Michael Potter, Peter Sullivan, and especially Warren Goldfarb. I presented some of the ideas in this paper at a Tractatus conference at Utrecht in 2000 and at an earlier Stirling Tractatus workshop in 2004. I have presented versions of this paper at the 2005 Stirling Tractatus conference, at the 2007 Leipzig Wittgenstein und Wissenschaft conference, and at colloquia at Carnegie Mellon University and Stanford University. Discussions on these occasions prompted improvements in the paper. I am especially grateful to Peter Sullivan for extensive comments on the penultimate version of this chapter.
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