The Foundations of Arithmetic

a logico-mathematical investigation into the concept of number[1]

[*Die Grundlagen der Arithmetik* was published in 1884. What follows here is the Introduction, §§1–4 (which further explain Frege's task), §§45–69 (which establish the philosophical foundations of Frege's logicist project), and §§87–91 and 104–9 (from the Conclusion). Summaries of the remaining sections are provided at the relevant points.]

I **Introduction**

If we ask what the number one is, or what the symbol 1 means,[2] we are more often than not given the answer: a thing. And if we then point out that the proposition

'The number one is a thing'

is not a definition, since it has the definite article on one side and the indefinite on the other, and that it only says that the number one belongs to the class of things, but not which thing it is, then we may well be invited to choose whatever we like to call the number one. But if everyone was allowed to understand by this name whatever he liked, then the same proposition about the number one would mean different things to different people; such propositions would have no common content. Some may reject the question, noting that the meaning of the

[1] Translated by Michael Beaney. Page numbers in the margin are from the original edition.

[2] Throughout this translation of *GL*, unless otherwise indicated, 'Bedeutung' and its cognates have been translated as 'meaning' and its cognates. On the translation of 'Bedeutung', see the Introduction, §4 above.

letter a in arithmetic cannot be given either; and if it were said: a means a number, then the same mistake would be made as in the definition: one is a thing. Now the rejection of the question in the case of a is quite justified: it means no particular, specifiable number, but serves instead to express the generality of propositions. If, in $a + a - a = a$, we sub-
II stitute for a any number we like, but the same throughout, | then a true equation is always obtained.[3] It is in this sense that the letter a is used. But in the case of one, the matter is essentially different. Can we, in the equation $1 + 1 = 2$, substitute for 1 both times the same object, say the Moon? It seems rather that we must substitute something different for the first 1 as for the second. Why is it that we must do here precisely what would be a mistake in the other case? Arithmetic does not manage with the letter a alone, but must also use other letters, b, c, etc., in order to express in general form relations between different numbers. So it might be supposed that the symbol 1 cannot be sufficient either, if it served in a similar way to confer generality on propositions. But does the number one not appear as a particular object with specifiable properties, e.g. that of remaining unchanged when multiplied by itself? In this sense, there are no properties of a that can be specified; since whatever is asserted of a is a common property of numbers, whereas $1^1 = 1$ asserts nothing of the Moon, nor of the Sun, nor of the Sahara, nor of the Peak of Tenerife; for what could the sense of such an assertion be?

To such questions not even a mathematician is likely to have a satisfactory answer ready to give. Yet is it not shameful that a science should be so unclear about its most prominent object, which is apparently so simple? Small wonder than no one can say what number is. If a concept that is fundamental to a great science poses difficulties, then it is surely an imperative task to investigate it in more detail and overcome these difficulties, especially since complete clarity will hardly be achieved concerning negative, fractional and complex numbers, so long as insight into the foundation of the whole structure of arithmetic is deficient. |
III Admittedly, many will not think this worth the trouble. This concept, they suppose, is quite adequately treated in the elementary textbooks

[3] 'Gleichung' has, throughout this volume, been translated as 'equation', which is what it unambiguously means. However, as noted above (p. 64, fn. 24), it is nevertheless clear that Frege understood 'Gleichheit' ('equality') in the sense of 'identity', and regarded equations as identities. (Cf. *BS*, §8 (pp. 64–5 above), where his symbol for 'Inhaltsgleichheit' was introduced; and *SB*, fn. A, p. 151 below.) It was this that led Austin to render 'Gleichung' as 'identity' in what is still the only complete translation of *GL* (see *FA*, p. II, fn.). But it is certainly more natural to call $1 + 1 = 2$, say, an *equation*, rather than an *identity*; and this has been respected here. Since Frege's primary concern in *GL* is obviously with arithmetic, 'Gleichheit' and 'gleich' too have normally been translated here as 'equality' and 'equal', although 'identity' and 'identical' have occasionally also been used (with the German term in square brackets following them) where they are clearly more appropriate.

and thereby settled once and for all. Who can then believe that he still has something to learn about so simple a matter? So free from any difficulty is the concept of positive whole number taken to be, that it is assumed that it can be explained scientifically and definitively to children, and that everyone, without further reflection or acquaintance with what others have thought, knows all about it. The first precondition for learning is thus frequently lacking: the knowledge that we do not know. The result is that we remain content with a crude conception, even though Herbart[A] has already provided a better one. It is depressing and discouraging that again and again an insight once achieved threatens to be lost in this way, and that so much work appears to be done in vain, because in our inflated conceit we do not think it necessary to appropriate its fruits. My work too, I am well aware, is exposed to such a danger. This crudity of conception surfaces when calculation is described as aggregative, mechanical thought.[B] I doubt that there is any such thought. Aggregative imagination there may well be; but that has no significance [*Bedeutung*] for calculation. Thought is essentially the same everywhere: it is not the case that there are different kinds of laws of thought depending on the object [of thought]. The differences [in thought] merely consist in the greater or lesser purity and independence from psychological
IV influences and external aids such as ordinary language, numerals | and suchlike, and also in the degree of refinement in the structure of concepts; but it is precisely in this respect that mathematics aims not to be surpassed by any other science, not even philosophy.

It will be seen from the present work that even an inference like that from n to $n + 1$, which is apparently peculiar to mathematics, is based on general logical laws, and that there is no need of special laws for aggregative thought. Admittedly, it is possible to manipulate numerals mechanically, just as it is possible to speak like a parrot; but that can scarcely be called thinking. It only becomes possible after mathematical symbolism has been so developed, through genuine thinking, that it does the thinking for us, so to speak. This does not show that numbers are formed in a particularly mechanical way, as sand, say, is formed from grains of quartz. It is in the interest of mathematicians, I think, to counter such a view, which is characterized by a disparagement of the principal object of their science and thereby that science itself. Yet even mathematicians are prone to say such things. Sooner or later, however, the concept of number must be recognized as having a finer structure than most of the concepts of other sciences, even though it is still one of the simplest in arithmetic.

[A] *Collected Works*, ed. Hartenstein, Vol. X, Part I, 'Umriss pädagogischer Vorlesungen', §252, fn. 2: 'Two does not mean [*heisst nicht*] two things, but doubling' etc.

[B] K. Fischer, *System der Logik und Metaphysik oder Wissenschaftslehre*, 2nd edn., §94.

In order, then, to dispel this illusion that no difficulties at all are posed by the positive whole numbers, but that general agreement prevails, it seemed to me a good idea to discuss some of the views of philosophers and mathematicians on the questions raised here. It will be seen how little accord is to be found, even outright contradictions emerging. Some say, for example, 'units are identical [*gleich*] with one another'; others hold that they are different; and both sides have reasons for their claim
V that cannot be rejected out of hand. Here I shall try | to motivate the need for a more exact investigation. At the same time, this preliminary elucidation of the views expressed by others will clear the ground for my own conception, by convincing people beforehand that these other paths do not lead to the goal, and that my opinion is not just one of many equally justified opinions; and so I hope to settle the question definitively, at least in essentials.

Admittedly, this has led me to take a more philosophical approach than many mathematicians may deem appropriate; but a fundamental investigation of the concept of number will inevitably turn out to be somewhat philosophical. The task is shared by mathematics and philosophy.

If the co-operation between these sciences, despite many attempts from both sides, is not as productive as might be wished or is surely possible, then this seems to me to be due to the prevalence of psychological modes of investigation, which have even penetrated logic. With this trend mathematics has no points of contact at all, and this easily explains the aversion of many mathematicians to philosophical investigations. When, for example, Stricker[C] calls the ideas of number motor phenomena, dependent on muscle sensations, no mathematician can recognize his numbers in this or knows where to begin with such a proposition. An arithmetic founded on muscle sensations would certainly be sensational, but it would also turn out to be just as vague as this foundation. No, arithmetic has nothing at all to do with sensations. Just as little has it to do with mental images, compounded from the traces of earlier sense impressions. The fluctuating and indeterminate nature
VI of these forms stands in stark contrast to the determinate and | fixed nature of mathematical concepts and objects. It may well be useful to investigate the ideas and changes of ideas that occur during mathematical thinking; but psychology should not suppose that it can contribute anything at all to the foundation of arithmetic. To the mathematician as such, these mental images, their origin and change are irrelevant. Stricker himself says that he associates nothing more than the idea of the symbol 100 with the word 'hundred'. Others may have the idea of the letter C or something else; does it not follow from this that these mental images are completely irrelevant and incidental to the essence

[C] *Studien über Association der Vorstellungen* (Vienna, 1883).

of the matter as it concerns us here, just as incidental as blackboard and chalk, and that they do not deserve to be called ideas of the number one hundred at all? The essence of the matter should not be seen to lie in such ideas! The description of the origin of an idea should not be taken for a definition, nor should the account of the mental and physical conditions for becoming aware of a proposition be taken for a proof, and nor should the discovery [*Gedachtwerden*] of a proposition be confused with its truth! We must be reminded, it seems, that a proposition just as little ceases to be true when I am no longer thinking of it as the Sun is extinguished when I close my eyes. Otherwise we would end up finding it necessary to take account of the phosphorous content of our brain in proving Pythagoras' theorem, and astronomers would shy away from extending their conclusions to the distant past, for fear of the objection: 'You reckon that $2 \times 2 = 4$ held then; but the idea of number has a development, a history! One can doubt whether it had reached that stage by then. How do you know that this proposition already existed at that point in the past? Might not the creatures living at that
VII time have held the proposition $2 \times 2 = 5$, from which | the proposition $2 \times 2 = 4$ only evolved through natural selection in the struggle for existence; and might not this in turn, perhaps, be destined in the same way to develop further into $2 \times 2 = 3$?' *Est modus in rebus, sunt certi denique fines!*[4] The historical mode of investigation, which seeks to trace the development of things from which to understand their nature, is certainly legitimate; but it also has its limitations. If everything were in continual flux and nothing remained fixed and eternal, then knowledge of the world would cease to be possible and everything would be thrown into confusion. We imagine, it seems, that concepts originate in the individual mind like leaves on a tree, and we suppose that their nature can be understood by investigating their origin and seeking to explain them psychologically through the working of the human mind. But this conception makes everything subjective, and taken to its logical conclusion, abolishes truth. What is called the history of concepts is really a history either of our knowledge of concepts or of the meanings of words. Often it is only through enormous intellectual work, which can last for hundreds of years, that knowledge of a concept in its purity is achieved, by peeling off the alien clothing that conceals it from the mind's eye. What are we then to say when someone, instead of carrying on this work where it still seems incomplete, ignores it entirely, and enters the nursery or takes himself back to the earliest conceivable stage of human development, in order there to discover, like John Stuart Mill, some gingerbread or pebble arithmetic! It remains only to ascribe to the flavour

[4] 'There is moderation in all things; there are, in short, fixed limits'; a quotation from Horace, *Satires*, Book I, I, line 106.

of the cake a special meaning for the concept of number. This is surely the exact opposite of a rational procedure and in any case as unmathematical as it could possibly be. No wonder that mathematicians want nothing to do with it! Instead of finding concepts in particular purity
VIII near to their imagined source, | everything is seen blurred and undifferentiated as through a fog. It is as though someone who wanted to learn about America tried to take himself back to the position of Columbus as he caught his first dubious glimpse of his supposed India. Admittedly, such a comparison proves nothing; but it does, I hope, make my point. It may well be that the history of discoveries is useful in many cases as preparation for further research; but it should not aspire to take its place.

As far as mathematicians are concerned, combatting such views would scarcely have been necessary; but since I wanted to resolve the disputed issues, as far as possible, for philosophers as well, I was forced to involve myself a little in psychology, if only to repel its incursion into mathematics.

Besides, psychological turns of phrase occur even in mathematical textbooks. If someone feels obliged to give a definition, and yet cannot do so, then he will at least describe the way in which the object or concept concerned is arrived at. This case is easily recognized by the absence of any further mention of such an explanation. For teaching purposes, such an introduction to things is quite in order; only it should always be clearly distinguished from a definition. A delightful example of how even mathematicians can confuse the grounds of proof with the mental or physical conditions for constructing proofs is afforded by E. Schröder,[D] in offering the following, under the heading 'Special Axiom': 'The intended principle could well be called the Axiom of the Inherence of Symbols. It gives us the assurance that in all our elaborations and infer-
IX ences the symbols | remain fixed in our memory – and even firmer on paper', etc.

Now just as much as mathematics must refuse any assistance from psychology, it must accept its close connection with logic. Indeed, I endorse the view of those who regard a sharp separation as impossible. It is at least granted that any investigation into the validity of a proof or the legitimacy of a definition must be logical. But such issues are not at all to be dismissed from mathematics, since it is only by resolving them that the necessary certainty is attained.

Admittedly, in this direction too I go somewhat further than is usual. Most mathematicians are content, in investigations of a similar kind, when they have satisfied their immediate needs. If a definition allows itself to be used in proofs, if contradictions are nowhere encountered,

[D] *Lehrbuch der Arithmetik und Algebra* [Leipzig, 1873].

if connections are revealed between apparently distant things, and if this yields greater order and regularity, then the definition is usually regarded as sufficiently established and few questions are asked about its logical justification. This procedure has in any case the advantage that it is unlikely entirely to fail in its purpose. I too think that definitions must show their worth by their fruitfulness, by their usefulness in constructing proofs. But it is well to observe that the rigour of a proof remains an illusion, however complete the chains of inference may be, if the definitions are only justified retrospectively, by the non-appearance of any contradiction. Fundamentally, then, only an empirical certainty is ever achieved, and it must really be accepted that in the end a contradiction might still be encountered that brings the whole edifice down in ruins. That is why I have felt obliged to go back somewhat further into the general logical foundations than most mathematicians, perhaps, would regard as necessary. |

X In this investigation I have adhered to the following fundamental principles:

There must be a sharp separation of the psychological from the logical, the subjective from the objective;
The meaning of a word must be asked for in the context of a proposition, not in isolation;
The distinction between concept and object must be kept in mind.

To comply with the first, I have used the word 'idea' ['*Vorstellung*'] always in the psychological sense, and have distinguished ideas from both concepts and objects. If the second principle is not observed, then one is almost forced to take as the meaning of words mental images or acts of an individual mind, and thereby to offend against the first as well. As concerns the third point, it is a mere illusion to suppose that a concept can be made into an object without altering it. From this it follows that a widely held formalist theory of fractional, negative numbers, etc., is untenable. How I intend to improve on it can be only indicated in the present work. In all these cases, as with the positive whole numbers, it will come down to fixing the sense of an equation.[5]

My results will, I think, at least in essentials, win the approval of those mathematicians who take the trouble to consider my arguments. They seem to me to be in the air, and separately they have, perhaps, already been stated, at least in rough form; though they may well be new in their connections with one another. I have sometimes been surprised that accounts that come so close to my conception on one point deviate so sharply on another.

The reception by philosophers will be varied, depending on their

[5] See §§62ff. (pp. 109ff. below).

XI standpoint; it will certainly be worst by | those empiricists who would recognize only induction as the original mode of inference, and even that not really as a mode of inference, but as habituation. One or another, perhaps, will take this opportunity to examine afresh the foundations of his theory of knowledge. To those who might want to declare my definitions unnatural, I would suggest that the question here is not whether they are natural, but whether they go to the heart of the matter and are logically unobjectionable.

I cherish the hope that even philosophers will find something useful in the present work, if they examine it without prejudice. |

* * *

1 **§1.** After departing for a long time from Euclidean rigour, mathematics is now returning to it, and even striving to take it further. In arithmetic, simply as a result of the origin in India of many of its methods and concepts, reasoning has traditionally been less strict than in geometry, which had mainly been developed by the Greeks. This was only reinforced by the discovery of higher analysis; since considerable, almost insuperable difficulties stood in the way of a rigorous treatment of this subject, whilst at the same time there seemed little profit in the expenditure of effort in overcoming them. Later developments, however, have shown more and more clearly that in mathematics a mere moral conviction, based on many successful applications, is insufficient. A proof is now demanded of many things that previously counted as self-evident. It is only in this way that the limits to their validity have in many cases been determined. The concepts of function, continuity, limit and infinity have been shown to require sharper definition. Negative and irrational numbers, which have long been accepted in science, have had to submit to a more exacting test of their legitimacy.

Thus everywhere efforts are being made to provide rigorous proofs, precise determinations of the limits of validity and, as a means to this, sharp definitions of concepts. |

2 **§2.** This path must eventually lead to the concept of Number[6] and the simplest propositions holding of the positive whole numbers, which

[6] I follow Austin here (cf. *FA*, p. 2, fn.) in translating 'Anzahl' by 'Number' (with a capital 'N'), leaving 'number' for the more general term 'Zahl'. The distinction plays little role in *GL* (cf. Frege's own fn. G below), but it does acquire significance in *GG*, II (anticipated at *GG*, I, §§41–2), when Frege distinguishes the real numbers ('reelen Zahlen') from the natural or cardinal numbers ('Anzahlen'), which are now to be understood as different from the positive whole numbers ('positiven ganzen Zahlen'). 'The natural numbers answer the question "How many objects of a certain kind are there?", whilst the real numbers can be regarded as measurement numbers [*Masszahlen*], which state how large a magnitude is compared with a unit magnitude' (*GG*, II, §157).

form the foundation of the whole of arithmetic. Admittedly, numerical formulae such as $5 + 7 = 12$ and laws such as that of the associativity of addition are so frequently confirmed by the countless applications that are made of them every day, that it can seem almost ludicrous to call them into question by demanding a proof. But it lies deep in the nature of mathematics always to prefer proof, wherever it is possible, to inductive confirmation. Euclid proved many things that would have been granted him anyway. And it was the dissatisfaction even with Euclidean rigour that led to the investigation of the Axiom of Parallels.[7]

Thus this movement towards ever greater rigour has already in many ways left behind the originally felt need, and the need has itself grown more and more in strength and extent.

The aim of proof is not only to place the truth of a proposition beyond all doubt, but also to afford insight into the dependence of truths on one another. After one has been convinced of the immovability of a boulder by vain attempts to shift it, the question then arises as to what secures it so firmly. The further these investigations are pursued, the fewer become the primitive truths to which everything is reduced; and this simplification is in itself a worthwhile goal. Perhaps the hope is even raised that, by bringing to light the general principles involved in what people have instinctively done in the simplest cases, general methods of concept-formation and justification may be discovered that will also be useful in more complicated cases. |

3 **§3.** Philosophical motives too have influenced my investigation. Questions as to the *a priori* or *a posteriori*, synthetic or analytic nature of arithmetical truths here await their answer. For even though these concepts themselves belong to philosophy, I still believe that no decision can be reached without help from mathematics. Admittedly, this depends on the sense that is given to the questions.

It frequently happens that we first discover the content of a proposition and then provide a rigorous proof in another, more difficult way, by means of which the conditions of its validity can often also be discerned more precisely. Thus in general the question as to how we arrive at the content of a judgement has to be distinguished from the question as to how we provide the justification for our assertion.

Now these distinctions between *a priori* and *a posteriori*, synthetic and analytic, in my opinion,[E] concern not the content of the judgement but the justification for making the judgement. Where there is no such justification, there is no possibility of drawing the distinctions either.

[E] By this I do not, of course, wish to introduce new senses, but only to capture what earlier writers, in particular *Kant*, have meant [*gemeint*]. [Cf. §§88–9, pp. 122–3 below.]

[7] For Frege's view of the Axiom of Parallels, see *EG*, pp. 251–2 below.

An *a priori* error is thus just as much an absurdity as, say, a blue concept. If a proposition is called *a posteriori* or analytic in my sense, then this is a judgement not about the psychological, physiological and physical conditions that have made it possible to form the content of the proposition in our mind, nor about how someone else, perhaps erroneously, has come to hold it to be true, but rather about the ultimate ground on which the justification for holding it to be true rests.

In this way the question is removed from the domain of psychology
4 and assigned to that of mathematics, if it | concerns a mathematical truth. It now depends on finding a proof and following it back to the primitive truths. If, on the way, only general logical laws and definitions are encountered, then the truth is analytic, assuming that propositions on which the admissibility of any definition rests are also taken into account. If it is not possible to provide a proof, however, without using truths that are not of a general logical nature, but belong instead to the domain of a particular science, then the proposition is synthetic. For a truth to be *a posteriori*, it must be impossible for its proof to avoid appeal to facts, that is, to unprovable and non-general truths that contain assertions about particular objects. If, on the other hand, it is possible to provide a proof from completely general laws, which themselves neither need nor admit of proof, then the truth is *a priori*.[F]

§4. Starting from these philosophical questions, we arrive at the same demand that had arisen independently in the domain of mathematics: that the fundamental theorems of arithmetic, wherever possible, must be proved with the greatest rigour; since only if the utmost care is taken to avoid any gaps in the chain of inference can it be said with certainty upon what primitive truths the proof is based; and only if these are known can the philosophical questions be answered. |

5 If an attempt is now made to meet this demand, then propositions are very soon encountered that cannot be proved so long as the concepts that occur in them cannot be analysed into simpler ones or the propositions cannot be reduced to something more general. Now here it is above all Number which must be defined or recognized as indefinable. This is the task of the present work.[G] On its outcome depends the decision as to the nature of arithmetical laws.

[F] If general truths are recognized at all, then it must also be granted that there are such primitive laws, since from purely individual facts nothing follows, except on the basis of a law. Even induction rests on the general proposition that this procedure can establish the truth or at any rate the probability of a law. For those who deny this, induction is nothing more than a psychological phenomenon, a way in which people come to believe in the truth of a proposition, without this belief thereby being at all justified.

[G] In what follows, therefore, unless otherwise indicated, no other numbers than the positive whole numbers will be under discussion, the numbers which answer the question 'how many?' [Cf. fn. 6 above.]

Before tackling these questions themselves, I shall first say something to provide a hint as to their answers. For if it should turn out that there are reasons, from other points of view, why the fundamental theorems of arithmetic are analytic, then this would also speak in favour of their provability and the definability of the concept of Number. Reasons for holding that these truths are *a posteriori* would have the opposite effect. The points at issue here may therefore first be submitted to a preliminary examination.

[The next forty sections (§§5–44; *GL*, pp. 5–58) are omitted here; but the following summary is offered of the main points, under the headings that Frege provides.

I. Views of certain writers on the nature of arithmetical propositions

Are numerical formulae provable? (§§5–8)

Frege argues against Kant that the lack of self-evidence of complex numerical formulae such as '135664 + 37863 = 173527' shows not that they are synthetic but that they are provable (§5). He agrees with Leibniz that even such simple formulae as '2 + 2 = 4' are provable via axioms and definitions, though he criticizes Leibniz's own proof for missing out the associative law. Defining every number in terms of its predecessor allows us to reduce the infinite set of numbers to the number one and the successor relation. (§6.) Frege argues against Mill's view that the truth of '3 = 2 + 1' depends on the empirical possibility of separating three objects, say, $^{0}{}_{0}{}^{0}$, into two parts, thus, oo o. It is just as well, Frege remarks, that not everything in the world is nailed down, for otherwise this separation could not be achieved, and 2 + 1 would not be 3! And what would be the physical facts underlying the numbers 0 and 1, or very large numbers? In fact, we can number more than just objects that we can physically separate: we can speak of three strokes of a clock, three sensations of taste, or three methods of solving an equation. (§7.) Frege accepts that we may require experience to *learn* the truths of arithmetic, but that does not make those truths 'empirical' as that term is used in opposition to 'a priori', since (as he stated in §3) the issue here concerns *justification*. (§8.)

Are the laws of arithmetic inductive truths? (§§9–11)

Frege argues here that Mill always confuses the applications of an arithmetical proposition with the pure proposition itself. That 2 unit volumes of liquid added to 5 unit volumes of liquid make 7 unit volumes of liquid only holds if the volume does not change as a result, say, of some chemical reaction; and '+', for example, does not mean a process of heaping up, since it can be applied in quite different situations. (§9.) Induction itself, if understood as involving judgements of probability, *presupposes* arithmetic. (§10.)

Are the laws of arithmetic synthetic *a priori* or analytic? (§§12–17)

Frege's definitions in §3 rule out the possibility of there being any analytic *a posteriori* truths, so if Mill's view that arithmetical truths are synthetic *a posteriori* is rejected, the only other possibilities are that they are synthetic *a priori*, as Kant thought, or analytic *a priori*. But in criticizing Kant, Frege remarks that it is all too easy to appeal to inner intuition when other grounds cannot be found. (§12.) Arithmetic is different from geometry (§13), which indeed contains synthetic truths. The basis of arithmetic lies deeper than that of either empirical science or geometry: 'The truths of arithmetic govern the realm of the numerable. This realm is the broadest; for to it belongs not only the actual, not only the intuitable, but everything thinkable. Should not the laws of number, then, stand in the most intimate connection with those of thought?' (§14.)[8] Frege endorses Leibniz's view that arithmetical propositions are analytic, though he recognizes that there is a sense in which *all* truths are 'analytic' for Leibniz (§15); and he quotes with approval Leibniz's remark that 'the concern here is not with the history of our discoveries, which is different for different people, but with the connection and natural order of truths, which is always the same' (§17; see Leibniz, *NE*, IV, vii, 9).

II. Views of certain writers on the concept of Number

Whilst, if Part I is right, arithmetical propositions may be provable, and every individual number greater than 1 definable in terms of its predecessor, this still leaves the status of the general laws governing proof unclear, and the number one itself and the successor relation to be defined. Frege discusses the number one in Part III; here he investigates the general concept of Number, since it is from this that the general laws are to be derived. (§§18–20.)

[8] Cf. 'Letter to Marty, 29.8.1882', p. 80 above.

Is Number a property of external things? (§§21–25)

Frege offers two reasons for not regarding numbers as properties such as solidity or colour. Firstly, such properties belong to external things independently of any choice of ours, whereas what Number we ascribe to something depends on our way of viewing it. The *Iliad*, for example, can be thought of as one poem, or as twenty-four Books, or as some large Number of verses; and a pile of cards can be thought of as one pack or as fifty-two cards. (§22.) *One* pair of boots can be thought of as *two* boots (§25). Secondly, number is applicable over a far wider range, being applicable, in particular, to what is *non*-physical, such as ideas, concepts and syllogistic figures (§24).

Is number something subjective? (§§26–27)

But this does not mean that number is subjective. Number is no less objective than, say, the North Sea, where there is also an element of human choice in determining its boundaries. Frege distinguishes what is *objective* (*objectiv*) from what is *actual* (*wirklich*), the latter being the *handleable* (*handgreiflich*) or *spatial* (*räumlich*), such that what is actual is only part of what is objective. Both the axis of the Earth and the centre of mass of the solar system are objective, but they are not actual like the Earth itself. What is objective is what is law-governed, conceivable and judgeable – independent of sensation, intuition and imagination, but not of reason, as Frege characterizes it. (§26.) Frege also objects to treating number as an idea, because this would make arithmetic psychology. 'If the number two were an idea, then it would straightaway be mine only. Another's idea is already, as such, another idea. We would then have perhaps many millions of twos. We would have to say: my two, your two, one two, all twos.' But there may then be not only, in some cases, many more numbers than we would normally countenance, but also, in other cases, none where they would be expected. '10^{10}', for example, might turn out to be an empty symbol, since there might be no being capable of having the appropriate idea. (§27.)[9]

Numbers as sets (§28)

Frege mentions one final theory, construing Numbers either as sets of objects or as sets of units. Neither view provides an account of the numbers 0 and 1; but his objections are clarified in Part III.

[9] Cf. *GG*, I, Preface, pp. XIV–XIX (pp. 201–6 below), which contains a more sustained attack on psychologism than Frege provides in *GL*, though the essential points remain.

III. Views on *Einheit* and *Eins*[10]

Does the number word 'one' express a property of objects? (§§29–33)

Further arguments are added to those offered in §§21–25 against viewing the number one, in particular, as a property of objects. Firstly, since 'oneness' would presumably be a property possessed by everything, describing something as 'one' would say nothing at all. 'Only through the possibility of something not being wise does the assertion that Solon is wise gain a sense. The content of a concept diminishes as its extension grows; if the latter becomes all-embracing, then the content must be lost entirely.' Secondly, if 'one' were a predicate, then 'Solon was one' would be just as legitimate as 'Solon was wise'. But 'Solon was one' is unintelligible on its own – without, say, 'wise man' being understood from the context. The point is even clearer in the plural case: 'Whilst we can combine "Solon was wise" and "Thales was wise" into "Solon and Thales were wise", we cannot say "Solon and Thales were one". The impossibility of this would not be perceived if "one" as well as "wise" were a property both of Solon and of Thales.' (§29.)

Are units identical with one another [*Sind die Einheiten einander gleich*]? (§§34–39)

Frege poses a dilemma for the view that numbers are sets. Either the things of which numbers are sets are different (as they would be if they were different objects), or else they are identical. If they are different, then there will be as many twos, say, as there are different pairs of objects in the universe. If they are identical (as talk of sets of 'units' would seem to suggest, supposedly abstracting away from all particular characteristics of objects), then (so to speak) they merge into one, and plurality is never attained. (§§34–39.) A distinction must be drawn between *unit* (*Einheit*) and *one* (*Eins*). 'Unit' is a concept word, whereas '1' is a proper name, and as such, does not admit of a plural. 'We say "the number one" and indicate by the definite article a definite and unique object of scientific inquiry. There are not different numbers one, but only one.' (§38.)

Attempts to overcome the difficulty (§§40–44)

Frege considers various attempts to resolve the problem of the supposed identity of 'units', by, amongst others, Jevons and Schröder, but finds them all wanting.

[10] Whilst 'Eins' clearly means 'one', 'Einheit' causes problems of translation, since it can mean 'unit' as well as 'unity' or 'oneness'. The ambiguity needs to be borne in mind in understanding Frege's arguments in this Part. Cf. fn. 18 below.

The translation resumes at the point where Frege begins to develop his positive account.]

58 Solution of the difficulty

§45. Let us now review what we have so far established and the questions that still remain unanswered.

Number is not abstracted from things in the way that colour, weight and hardness are, and is not a property of things in the sense that they are. The question still remains as to what it is of which something is asserted in making a statement of number [*Zahlangabe*].[11]

Number is not anything physical, but nor is it anything subjective, an idea.

Number does not result from the adding of thing to thing. Even naming each addition does not alter the situation.

The expressions 'multitude', 'set', 'plurality', due to their vagueness, are unsuitable for use in defining number.

With regard to one [*Eins*] and unity [*Einheit*], the question remains as to how the element of choice in our conceptions, which seems to blur every distinction between one and many, is to be restricted.

Distinguishability, indivisibility, unanalysability cannot be taken as marks[12] of what we express by the word 'one'.

If the things to be numbered are called units, then the unconditional assertion that units are identical [*gleich*] is false. That they are identical in certain respects is no doubt correct but worthless. The difference between the things to be numbered is actually necessary if the number is to be greater than 1.

It thus seems that we must ascribe two contradictory properties to units: identity [*Gleichheit*] and distinguishability.

A distinction must be drawn between one [*Eins*] and unit [*Einheit*].
59 The word 'one', as the proper name of an object | of mathematical inquiry, does not admit of a plural. It therefore makes no sense to let numbers result from the combination of ones. The plus sign in 1 + 1 = 2 cannot mean such a combination.

§46. To throw light on the matter, it will help to consider number in the context of a judgement that brings out its ordinary use. If, in looking at the same external phenomenon, I can say with equal truth 'This is a copse' and 'These are five trees', or 'Here are four companies' and 'Here are 500 men', then what changes here is neither the individual nor the whole, the aggregate, but rather my terminology. But that is

[11] On the translation of this term, see fn. 13 below.

[12] On Frege's use of the term 'Merkmal', see §53 (pp. 102–3 below); *CO*, pp. 189–90 below.

only a sign of the replacement of one concept by another. This suggests as the answer to the first question of the previous section that a statement of number contains an assertion about a concept.[13] This is perhaps clearest in the case of the number 0. If I say 'Venus has 0 moons', then there is no moon or aggregate of moons to assert anything of at all; but instead it is the concept 'moon of Venus' to which a property is ascribed, namely, that of including nothing under it. If I say 'The King's carriage is drawn by four horses', then I am ascribing the number four to the concept 'horse that draws the King's carriage'.[14]

It may be objected that a concept such as 'inhabitant of Germany', even though its marks remain the same, would have a property that changed from year to year, if a statement of number did assert something about it. It is fair to reply that objects too change their properties without preventing us from recognizing them as the same. But here there is a more particular explanation. For the concept 'inhabitant of Germany' contains the time as a variable component, or, to put it math-
60 ematically, | is a function of the time. Instead of '*a* is an inhabitant of Germany', we can say '*a* inhabits Germany', and this relates to the present point in time. Thus there is already something fluid in the concept itself. On the other hand, the same number belongs to the concept 'inhabitant of Germany at the beginning of the year 1883, Berlin time' throughout eternity.

§47. That a statement of number expresses something factual independent of our conceptions can only surprise those who regard a concept as something subjective like an idea. But this view is wrong. If, for

[13] The translation of this key Fregean thesis – '*die Zahlangabe enthält eine Aussage von einem Begriffe*' – has generated some controversy. For Frege, a 'Zahlangabe' answers the question 'How many?' (cf. fn. 6 and Frege's fn. G above), and takes the form 'There are *n F*'s', and 'statement of number' (perhaps most literally, 'giving of a number') is as good a translation as any. 'Aussage' has been more problematic, since 'assertion', which is what it usually means, is generally used to identify a certain kind of *speech act*, to be distinguished from questions, commands, etc.; whereas, it has been argued, there are clearly legitimate uses of number terms in asking, say, 'Are there five plates on the table?', or ordering, say, 'Put your two knives on the floor!' Dummett has suggested that the thesis is best rendered as 'The content of an ascription of number consists in predicating something of a concept' (1991a: p. 88); but the conciseness of Frege's own formulation is then lost. In any case, when we do make a *statement* of number (i.e. say 'There are *n F*'s'), we can indeed be seen as *asserting* something about a concept, and there is nothing in this that implies that questions, say, cannot be asked involving number terms. Frege just happens to have restricted his thesis to statements (where there is a truth-value to be ascribed); and the question as to what to say in the case of other speech acts – how to generalize the thesis – is left open. Frege's key point, then, is simply that, in answering the question 'How many?', we are saying something about a concept; and the most accurate, as well as the most concise, translation of his thesis is therefore the one adopted here – to be understood, though, in the light of what has just been said.

[14] The verb translated twice here as 'ascribe' is 'beilegen': see fn. 16 below.

example, we subordinate the concept of body to the concept of what has weight, or the concept of whale to the concept of mammal, then we are thereby asserting something objective. Now if the concepts were subjective, then the subordination of one to the other, as a relation between them, would also be subjective, just as a relation between ideas is. Admittedly, at first sight the proposition

'All whales are mammals'

appears to be about animals, not concepts; but if it is asked which animal is then being spoken of, there is no single one that can be picked out. Even assuming that a whale is present, our proposition still asserts nothing about it. We cannot infer from it that the animal present is a mammal, without the additional proposition that it is a whale, as to which our proposition says nothing. In general, it is impossible to speak of an object without in some way designating or naming it. But the word 'whale' does not name any individual creature. If it be replied that an individual, definite object is certainly not what is being spoken of, but rather an indefinite one, then I suspect that 'indefinite object' is only another expression for 'concept', and a poorer, self-contradictory
61 one at that. | Even if our proposition can only be justified by observing individual animals, this proves nothing as to its content. Whether it is true or not, or on what grounds we hold it as true, is irrelevant to the question as to what the proposition is about. If, then, a concept is something objective, then an assertion about it can also contain something factual.

§48. The false impression given by some earlier examples that different numbers may belong to the same thing is explained by the fact that objects were there taken as the bearers of number. As soon as we restore to its rightful place the true bearer, the concept, numbers reveal themselves as just as mutually exclusive in their realm as colours are in theirs.

We now also see how number can come to be thought of as arrived at by abstraction from things. What is actually obtained is a concept, in which the number is then discovered. Thus abstraction often does, in fact, precede the formation of a judgement of number. The confusion is the same as if it were said: the concept of fire risk is obtained by building a half-timbered house with wooden gables, thatched roof and draughty chimneys.

The power of collecting together that a concept has far surpasses the unifying power of synthetic apperception. By means of the latter it would not be possible to combine the inhabitants of Germany into a whole; but they can certainly be brought under the concept 'inhabitant of Germany' and counted.

The extensive applicability of number can now also be explained. It is indeed puzzling how the same can be asserted of physical and mental phenomena alike, of the spatial and temporal as well as of the non-spatial and non-temporal. But this is not at all what happens in state-
62 ments of number. Only concepts, under which the physical | and mental, the spatial and temporal, the non-spatial and non-temporal are brought, are ascribed numbers.

§49. We find confirmation of our view in Spinoza, who says:[H] 'I answer that a thing is called one or single merely with respect to its existence, and not its essence; for we conceive of things in terms of number only after they have been brought under a common measure. For example, whoever holds in his hand a sesterce and a dollar will not think of the number two unless he can give this sesterce and this dollar one and the same name, viz. piece of silver or coin; then he can affirm that he has two pieces of silver or coins; since he designates by the name coin not only the sesterce but also the dollar.' When he goes on: 'From this it is clear that a thing is called one or single only after another thing has been conceived that (as has been said) agrees with it', and when he thinks that God cannot be called one or single in any real sense, because we can form no abstract concept of his essence, then he goes wrong in thinking that a concept can only be acquired directly by abstraction from particular objects. A concept can just as well be acquired via its marks; and then it is possible for nothing to fall under it. If this did not happen, we would never be able to deny existence, and hence the affirmation of existence would lose its content too.

§50. E. Schröder[I] emphasizes that, to be able to speak of the frequency of a thing, the name of this thing must always be a *generic term*, a general concept word (*notio communis*): 'For as soon as an object is
63 pictured completely – with all | its properties and relations, it will stand out in the world as unique and its like will no longer be found. The name of the object then takes on the character of a *proper name* (*nomen proprium*) and the object cannot be thought of as one that occurs anywhere else. But this holds not only of *concrete* objects; it holds in general of anything, even where the idea of it arises through *abstractions*, provided only that this idea contains in it sufficient elements to fully determine the thing concerned . . . [Becoming an object that can be counted] is only possible for a thing in so far as one disregards or *abstracts from* some of its characteristic marks and relations, which distinguish it from

[H] Baumann, *Die Lehren von Zeit, Raum und Mathematik* [Berlin, 1868], Vol. I, p. 169. [Most of Frege's quotations from other writers are taken from this edited collection. The original work in this case is Spinoza's *Epistolae doctorum quorundam virorum*, No. 50.]

[I] Op. cit., p. 6.

all other things, by means of which the name of the thing then becomes a concept applicable to more things.'

§51. The truth in this account is clothed in such distorted and misleading language that it has to be disentangled and sifted out. First of all, it will not do to call a general concept word the name of a thing. The illusion then arises that number is a property of things. A general concept word just designates a concept. Only with the definite article or a demonstrative pronoun does it function as a proper name of a thing, but it then ceases to function as a concept word. The name of a thing is a proper name. An object does not occur anywhere else, but several objects may fall under a concept. That a concept is not only obtained by abstraction from the things that fall under it has already been noted in connection with Spinoza. Here I will add that a concept does not cease to be a concept when only one single thing falls under it, which thing is therefore completely determined by it. It is just that what belongs to such a concept (e.g., satellite of the Earth) is the num-
64 ber one, | which is a number in the same sense as 2 and 3. With a concept the question is always whether anything, and if so what, falls under it. With a proper name such questions make no sense. One should not be deceived by the use in language of a proper name, e.g. Moon, as a concept word, and vice versa; the distinction nevertheless remains. As soon as a word is used with the indefinite article or in the plural without an article, it is a concept word.

§52. Further confirmation of the view that number is ascribed to concepts can be found in our ordinary use of language, in saying ten man, four mark, three barrel.[15] The use of the singular here may indicate that the concept is intended, not the thing. The advantage of this form of expression is particularly evident in the case of the number 0. Elsewhere, admittedly, language ascribes number to objects, not to concepts: we say 'number of bales' just as we say 'weight of bales'. Thus we are apparently speaking of objects, whereas in truth we intend to assert something of a concept. This use of language is confusing. The expression 'four thoroughbred horses' generates the illusion that 'four' qualifies the concept 'thoroughbred horse' just as 'thoroughbred' qualifies the concept 'horse'. However, only 'thoroughbred' is such a mark; we use the word 'four' to assert something of a concept.

§53. By properties that are asserted of a concept I do not, of course, mean [*verstehe*] the marks that make up the concept. These are properties

[15] This is a direct translation of the German. In English we might talk, for example, of a ten man crew, a five pound note or a three barrel consignment.

of the things that fall under the concept, not of the concept. Thus 'right-angled' is not a property of the concept 'right-angled triangle'; but the proposition that there is no right-angled, rectilinear, equilateral triangle does express a property of the concept 'right-angled, rectilinear, equilateral triangle'; it ascribes to this the number zero. |

65 In this respect existence is similar to number. Affirmation of existence is indeed nothing other than denial of the number zero. Since existence is a property of concepts, the ontological proof of the existence of God fails in its aim.[16] But oneness [*Einzigkeit*] is just as little a mark of the concept 'God' as existence. Oneness cannot be used to define this concept any more than strength, spaciousness and homeliness can be used together with stones, mortar and beams to build a house. However, it should not be concluded that a property of a concept can never be deduced from the concept, that is, from its marks. Under certain circumstances this is possible, just as we can occasionally infer the durability of a building from the type of stone. It would therefore be going too far to assert that oneness or existence can never be inferred from the marks of a concept; it is just that this can never happen as directly as the mark of a concept can be ascribed as a property to an object that falls under the concept.

It would also be wrong to deny that existence and oneness can ever be marks of concepts. They are just not marks of concepts in which language suggests they are included. If, for example, all concepts under which only one object falls, are collected under one concept, then oneness is a mark of this concept. Under it would fall, for example, the concept 'moon of the Earth', though not the heavenly body itself. Thus a concept can fall under a higher one, that is to say, a concept of second order. But this relationship is not to be confused with that of subordination.[17]

[16] We should, perhaps, note here that it is potentially misleading to say that existence is a property of concepts, just as Frege himself later warns us against calling numbers themselves properties of concepts (see §57). Rather, affirming the existence of something is to be understood as attributing to the relevant concept the property of *being instantiated* (e.g. to say that God exists is to say that the concept *God* falls under the [second-level] concept *is instantiated*), just as saying that there are *n F*'s is to be understood as attributing to the concept *F* the property of *being instantiated n-fold*. To talk, in the latter case, of 'ascribing' numbers to concepts is not to be confused with 'ascribing' properties to things. Perhaps we should distinguish the 'ascription' of numbers from the 'attribution' of properties. Unfortunately, Frege himself uses one word – 'beilegen' – for both (e.g. in §46); so it seems best just to note the potential confusions here, rather than ascribe to Frege himself a more subtle distinction.

[17] In other words, 'The concept *moon of the Earth* is a concept under which only one object falls' asserts a relationship between a first-level and a second-level concept (as Frege later calls them; see esp. *CO*, pp. 183, 189–90 below), whereas 'All whales are mammals' asserts that the concept *whale* is subordinate to the concept *mammal* (cf. §47). Cf. also *NLD*, p. 364 below.

§54. It now becomes possible to give a satisfactory account of units. E. Schröder says on p. 7 of his textbook cited above: 'This generic term
66 or concept | will be called the denomination [*Benennung*] of the number formed in the way indicated and constitutes the essence of its unit'.

In fact, would it not be most appropriate to call a concept the unit that relates to the Number which belongs to it?[18] We can then give a sense to assertions that are made about the unit, that it is separated from its surroundings and indivisible. For the concept to which the number is ascribed does in general delimit what falls under it in a definite way. The concept 'letter in the word "Zahl"' delimits the *Z* from the *a*, the *a* from the *h*, and so on. The concept 'syllable in the word "Zahl"' picks out the word as a whole and as indivisible in the sense that the parts do not now fall under the concept. Not all concepts work this way. We can, for example, divide up what falls under the concept 'red' in a variety of ways, without the parts ceasing to fall under it. To such a concept no finite number belongs. The proposition concerning the distinguishability and indivisibility of units can therefore be stated thus:

Only a concept that delimits what falls under it in a definite way and allows no arbitrary division [of what falls under it] into parts[19] can constitute the unit that relates to a finite Number.

It will be noticed, however, that indivisibility here has a special meaning.

We can now easily answer the question as to how the identity of units

[18] 'Einheit' again causes problems of translation here. Schröder talks of the concept constituting 'das Wesen ihrer Einheit', and Frege is clearly picking up on this idea in suggesting that we call a concept 'Einheit . . . in Bezug auf die Anzahl, welche ihm zukommt', i.e. that we identify a concept with the 'unitness' that enables us to regard the objects that fall under the concept as its 'units'. The ambiguity of the word 'Einheit' is rather lost in translating it simply as 'unit'. A concept is not itself a unit but instead constitutes the condition that has to be met for an object to be its unit. But perhaps with the definite article, the term 'the unit' has a similar degree of ambiguity in English, referring either to a particular object or else to the concept by means of which objects are to be sorted ('the unit of assessment'). Compare, for example, 'The horse is a noble creature', which can be construed as being either about a particular horse ('The horse in the stable over there is a noble creature') or about the concept *horse* ('All horses are noble creatures' or 'The concept *horse* is subordinate to the concept *noble creature*'). In the latter sense we might perhaps talk of the concept being *the unit* – by means of which objects are to be sorted and hence numbered. And as Frege goes on to argue in this section, we must indeed distinguish between the unit *qua* individual object (which makes units different) and the unit *qua* instantiation of a concept (which makes units the same); though, strictly speaking, even this is not quite how Frege himself puts it, since in the latter case, we might still be taken as referring to objects, which can just be regarded in a certain way (as all falling under the same concept) – whereas what is actually identical across the cases is the concept itself.

[19] I.e. what is now called a *sortal* concept – a concept that *sorts* into particulars the objects that fall under it, which can then be counted.

is to be reconciled with their distinguishability. The word 'unit' is being used here in a double sense. Units are identical if the word has the meaning explained above. In the proposition 'Jupiter has four moons', the unit is 'moon of Jupiter'. Under this concept falls moon I as well as moon II, moon III and moon IV. Thus we can say: the unit to which I relates is identical with the unit to which II relates, and so on. Here
67 we have identity. But if it is the distinguishability | of units that is asserted, then by this is understood the distinguishability of the things numbered.

IV. The concept of Number

Every individual number is an independent object

§55. Having recognized that a statement of number contains an assertion about a concept, we can attempt to complete the Leibnizian definitions of the individual numbers by defining 0 and 1.

It is natural to say: the number 0 belongs to a concept if no object falls under it. But this appears to replace 0 by 'no', which means the same. The following formulation is therefore preferable: the number 0 belongs to a concept if, whatever *a* may be, the proposition holds universally that *a* does not fall under that concept.

In a similar way we could say: the number 1 belongs to a concept *F* if, whatever *a* may be, the proposition does not hold universally that *a* does not fall under *F*, and if from the propositions

'*a* falls under *F*' and '*b* falls under *F*'

it follows universally that *a* and *b* are the same.

It still remains to give a general definition of the transition from one number to the next. We could try the following formulation: the number $(n + 1)$ belongs to the concept *F* if there is an object *a* falling under *F* such that the number *n* belongs to the concept 'falling under *F*, but not *a*'.[20]

[20] In modern notation, using the device of the *numerical quantifier*, '$\exists_n x$' being read as 'there are *n* *x*'s such that', the three definitions here can be formalized thus:

(F_0) '$(\exists_0 x)Fx$' is defined as '$(\forall x)\ \neg Fx$'.
(F_1) '$(\exists_1 x)Fx$' is defined as '$\neg(\forall x)\ \neg Fx\ \&\ (\forall x)(\forall y)(Fx\ \&\ Fy \rightarrow x = y)$'.
(F_{n+1}) '$(\exists_{n+1} x)Fx$' is defined as '$(\exists x)[Fx\ \&\ (\exists_n y)(Fy\ \&\ x \neq y)]$'.

What this shows, of course, is that number statements of the form 'The number *n* belongs to a concept *F*' can indeed be logically defined. The objection that Frege goes on to raise is not that these definitions are wrong, but that they are, as they stand, insufficient to determine what numbers are.

§56. These definitions offer themselves so naturally after our previous results that an explanation is required as to why they cannot satisfy us.

The last definition is the most likely to raise doubts; for strictly
68 speaking the sense of the expression | 'the number n belongs to the concept G' is just as unknown to us as that of the expression 'the number $(n + 1)$ belongs to the concept F'. We can, of course, by means of this and the second definition say what is meant by

'the number $1 + 1$ belongs to the concept F',

and then, using this, give the sense of the expression

'the number $1 + 1 + 1$ belongs to the concept F',

and so on; but we can never – to take an extreme example – decide by means of our definitions whether the number *Julius Caesar* belongs to a concept, or whether that well-known conqueror of Gaul is a number or not. Furthermore, we cannot prove with the help of our attempted definitions that if the number a belongs to the concept F and the number b belongs to the same concept, then necessarily $a = b$. The expression '*the* number that belongs to the concept F' could not therefore be justified and it would thus be quite impossible to prove a numerical equality, since we would be unable to apprehend a definite number at all. It is only an illusion that we have defined 0 and 1; in truth we have only determined the sense of the phrases

'the number 0 belongs to',
'the number 1 belongs to';

but this does not allow us to distinguish 0 and 1 here as independent, reidentifiable objects.

§57. This is the place to gain a clearer understanding of our thesis that a statement of number contains an assertion about a concept. In the proposition 'The number 0 belongs to the concept F', 0 is only a part of the predicate, if the concept F is taken as the real subject.[21] I have therefore avoided calling a number such as 0, 1 or 2 a *property* of a concept. The individual number, by forming only a part of the predicate, appears precisely as an independent object. I have already remarked above that we say 'the number 1' and use the definite article
69 to register 1 as an object. | This independence manifests itself throughout arithmetic – as, for example, in the equation $1 + 1 = 2$. Since what

[21] I.e. rewriting the proposition as 'The concept F is ascribed the number 0', 'the number 0' is only a part of the predicate 'is ascribed the number 0'.

concerns us here is to define a concept of number that is useful for science, we should not be put off by the attributive form in which number also appears in our everyday use of language. This can always be avoided. For example, the proposition 'Jupiter has four moons' can be converted into 'The number of Jupiter's moons is four'. Here the 'is' should not be taken as a mere copula, as in the proposition 'The sky is blue'. This is shown by the fact that one can say: 'The number of Jupiter's moons is the number 4'. Here 'is' has the sense of 'is equal to', 'is the same as'. We thus have an equation that asserts that the expression 'the number of Jupiter's moons' designates the same object as the word 'four'. And equations are the prevalent form of proposition in arithmetic. It is no objection to this account that the word 'four' contains nothing about Jupiter or moons. There is also nothing in the name 'Columbus' about discovery or America and yet it is the same man who is called both Columbus and the discoverer of America.

§58. It might be objected that we can form no idea[J] at all of the object that we are calling four or the number of Jupiter's moons as something independent. But it is not the independence that we have granted to number that is to blame. It is very easy to think that in the idea of four spots on a die there is something that corresponds to the word 'four'; but that is an illusion. Imagine a green meadow and test whether the idea changes when the indefinite article is replaced by the number word 'one'. Nothing happens, whereas something does correspond in the
70 idea to the word 'green'. | If we picture the printed word 'gold', we do not at first think of any number in doing so. If we now ask ourselves how many letters it contains, then the result is the number 4; but the idea does not thereby become any more definite, but may remain quite unchanged. We only discover the number on the introduction of the concept 'letter in the word "gold"'. In the case of the four spots on a die, the matter is somewhat obscured, since the concept springs so immediately to mind, due to the similarity of the spots, that we hardly notice its intervention. The number can be pictured neither as an independent object nor as a property in an external thing, since it is neither something sensible nor a property of an external thing. The matter is certainly clearest in the case of the number 0. One will try in vain to picture 0 visible stars. One may well imagine the sky completely clouded over; but there is nothing in this that corresponds to the word 'star' or to 0. One only pictures a situation that prompts the judgement: there is now no star to be seen.

§59. Every word, perhaps, evokes some idea in us, even such a word as 'only'; but the idea need not correspond to the content of the word;

[J] 'Idea' taken in the sense of something pictorial.

it may be quite different in different people. One may well picture here a situation which invites a proposition in which the word occurs; or the spoken word may call to mind the written word.

This does not only happen in the case of particles. There is certainly no doubt that we cannot form any idea of our distance from the Sun. For even though we know the rule concerning how many measuring rods must be laid end to end, we still fail in every attempt to sketch a picture, according to this rule, that even only faintly approximates to
71 what we want. But that is no reason to doubt the | correctness of the calculation which determined the distance, and it in no way prevents us from basing further inferences on the existence of this distance.

§60. Even so concrete a thing as the Earth cannot be pictured as we know it to be; but we content ourselves with a ball of moderate size, which serves us as a symbol for the Earth; yet we realize that this is very different from it. Now even though our idea often fails at all to capture what we want, we still make judgements about an object such as the Earth with great confidence, even where its size is at issue.

We are quite often led by our thought beyond the imaginable, without thereby losing the support for our inferences. Even if, as it seems to be, it is impossible for us as human beings to think without ideas, it may still be that their connection with thought is entirely inessential, arbitrary and conventional.

That no idea can be formed of the content of a word is therefore no reason for denying it any meaning or for excluding it from use. The appearance to the contrary doubtless arises because we consider the words in isolation and in asking for their meaning look only for an idea. A word for which we lack a corresponding mental picture thus appears to have no content. But one must always keep in mind a complete proposition. Only in a proposition do the words really have a meaning.[22] The mental pictures that may pass before us need not correspond to the logical components of the judgement. It is enough if the proposition as a whole has a sense; its parts thereby also obtain their content.

This observation, it seems to me, is likely to throw light on a good
72 many | difficult concepts, such as that of the infinitesimal,[K] and its implications are certainly not restricted to mathematics.

The independence that I am claiming for number is not to be taken

[K] It all depends on defining the sense of an equation of the form

$$df(x) = g(x)dx,$$

rather than showing that there is a line bounded by two distinct points whose length is dx.

[22] This marks Frege's first use of the context principle in *GL*. For discussion of the role of the context principle in Frege's philosophy, see the Introduction, pp. 15–20 above.

to mean that a number word designates something when not in the context of a proposition, but I only intend by this to exclude the use of a number word as a predicate or attribute, which rather changes its meaning.

§61. But, it may perhaps be objected, even if the Earth cannot really be pictured, it is still an external thing, which has a definite location; but where is the number 4? It is neither outside us nor in us. In the spatial sense, that is certainly true. Fixing the location of the number 4 makes no sense; but it follows from this only that it is not a spatial object, not that it is not an object at all. Not every object is somewhere. Even our ideas[L] are in this sense not in us – under our skin. Here there are ganglion cells, blood corpuscles and suchlike, but not ideas. Spatial predicates are not applicable to them: one idea is neither to the right nor to the left of another; there are no distances between ideas measurable in millimetres. If we nevertheless speak of them as in us, then we mean by this that they are subjective.

But even if [we admit that] what is subjective has no location, how is it possible for the number 4, which is objective, not to be anywhere? Now I maintain that there is no contradiction at all in this. The number 4 is, in fact, exactly the same for everyone who deals with it; but this has nothing to do with being spatial. Not every objective object [*objectives Gegenstand*] has a location. |

73 To obtain the concept of Number, the sense of a numerical equation must be determined

§62. How, then, is a number to be given to us, if we cannot have any idea or intuition of it? Only in the context of a proposition do words mean something. It will therefore depend on defining the sense of a proposition in which a number word occurs. As it stands, this still leaves much undetermined. But we have already established that number words are to be understood as standing for independent objects. This gives us a class of propositions that must have a sense – propositions that express recognition [of a number as the same again]. If the symbol *a* is to designate an object for us, then we must have a criterion that decides in all cases whether *b* is the same as *a*, even if it is not always in our power to apply this criterion. In our case we must define the sense of the proposition

> 'The number that belongs to the concept *F* is the same as the number that belongs to the concept *G*';

[L] Understanding this word purely psychologically, not psychophysically.

that is, we must represent the content of this proposition in another way, without using the expression

'the Number that belongs to the concept F'.

In doing so, we shall be giving a general criterion for the equality of numbers. When we have thus acquired a means of grasping a definite number and recognizing it as the same again, we can give it a number word as its proper name.

§63. Hume[M] has already mentioned such a means: 'When two numbers are so combined, as that the one has always a unit answering to
74 every unit of the other, we pronounce them equal'. The | view that equality of numbers must be defined in terms of one-one correlation[23] seems recently to have gained widespread acceptance amongst mathematicians.[N] But it initially raises logical doubts and difficulties, which we ought not to pass over without examination.

The relationship of equality [*Gleichheit*] does not hold only amongst numbers. From this it seems to follow that it ought not to be defined specially for this case. One would think that the concept of equality would already have been fixed, from which, together with the concept of Number, it must then follow when Numbers are equal to one another, without requiring any further, special definition.

Against this, it is to be noted that for us the concept of Number has not yet been fixed, but is only to be determined by means of our definition. Our aim is to form the content of a judgement that can be construed as an equation on each side of which is a number. We thus do not intend to define equality specially for this case, but by means of the concept of equality, taken as already known, to obtain that which is to be regarded as being equal. Admittedly, this seems to be a very unusual kind of definition, which has certainly not yet received sufficient attention from logicians; but that it is not unheard of may be shown by a few examples.

§64. The judgement 'Line a is parallel to line b', in symbols:

$$a \,//\, b,$$

[M] Baumann, op. cit., Vol. II, p. 565 [Hume, *A Treatise of Human Nature*, Book I, Part III, §1, p. 71].

[N] Cf. E. Schröder, op. cit., pp. 7–8; E. Kossak, *Die Elemente der Arithmetik, Programm des Friedrichs-Werder'schen Gymnasiums* (Berlin, 1872), p. 16; G. Cantor, *Grundlagen einer allgemeinen Mannichfaltigkeitslehre* (Leipzig, 1883).

[23] Frege actually uses the phrase 'eindeutige Zuordnung', by which he means a *many-one* relation, 'beiderseits eindeutige Zuordnung' being what he calls *one-one* correlation, i.e. a relation that is both many-one and one-many (see p. 77 above). But it seems more natural to talk of the latter here.

can be construed as an equation. If we do this, we obtain the concept of direction and say: 'The direction of line a is equal to the direction
75 of line b'. | We thus replace the symbol // by the more general =, by distributing the particular content of the former to a and b. We split up the content in a different way from the original way and thereby acquire a new concept. Admittedly, the process is often seen in reverse, and parallel lines are frequently defined as lines whose directions are equal. The proposition 'If two lines are parallel to a third, then they are parallel to one another' can then very easily be proved by appealing to the corresponding proposition concerning equality [of directions]. It is only a pity that this stands the true situation on its head! For everything geometrical must surely originate in intuition. I now ask whether anyone has had an intuition of the direction of a line. Of the line, certainly! But is the direction of a line distinguished in intuition from the line itself? Hardly! This concept [of direction] is only found through a mental act that takes off from intuition. On the other hand, one does have an idea of parallel lines. The proof just mentioned only works by covertly presupposing, in the use of the word 'direction', what is to be proved; for were the proposition 'If two lines are parallel to a third, then they are parallel to one another' false, then $a \,//\, b$ could not be transformed into an equation.

Similarly, from the parallelism of planes, a concept can be obtained that corresponds to that of direction in the case of lines. I have seen the word 'orientation' ['*Stellung*'] used for this. From geometrical similarity there arises the concept of shape, so that, for example, instead of 'The two triangles are similar', one says: 'The two triangles have equal shapes' or 'The shape of the one triangle is equal to the shape of the other'. So too, from the collinear relationship of geometrical figures, a concept can be obtained for which a name has still to be found. |

76 **§65.** Now in order to get, for example, from parallelism[O] to the concept of direction, let us try the following definition: the proposition

'Line a is parallel to line b'

is to mean the same as

'The direction of line a is equal to the direction of line b'.

This definition is unusual inasmuch as it apparently specifies the already known relation of equality, whereas it is actually intended to

[O] To express myself more easily and to be more readily understood, I take here the case of parallelism. The essentials of the discussion can be readily carried over to the case of numerical equality.

introduce the expression 'the direction of line *a*', which only occurs incidentally. From this there arises a second doubt, as to whether such a definition might not involve us in conflict with the well-known laws of identity [*Gleichheit*].[24] What are these? As analytic truths, they should be derivable from the concept itself. Leibniz[P] offers the following definition:

'*Eadem sunt, quorum unum potest substitui alteri salva veritate*'.[25]

I shall adopt this definition of identity [*Gleichheit*] as my own. Whether one says 'the same' ['*dasselbe*'], like Leibniz, or 'equal' ['*gleich*'], is unimportant. 'The same' may appear to express complete agreement, 'equal' only agreement in this or that respect; but a form of words can be employed in which this distinction ceases to apply: instead of 'The lines are equal in length', for example, one can say 'The length of the lines is equal' or 'the same'; instead of 'The surfaces are identical [*gleich*] in colour', one can say 'The colour of the surfaces is identical [*gleich*]'.[26]
77 And this is the way we used the word in the examples above. | In universal substitutability, in fact, all the laws of identity [*Gleichheit*] are contained.

In order to justify our suggested definition of the direction of a line, we would thus have to show that

'the direction of *a*'

can be everywhere substituted by

'the direction of *b*',

[P] *Non inelegans specimen demonstrandi in abstractis* (Erdmann edn. [*Oper. Philos.* I], p. 94).

[24] Here is one occasion on which the translation of 'Gleichheit' as 'identity' rather than 'equality', which Frege goes on to indicate he treats as synonymous, is more appropriate.

[25] 'Those things are the same of which one can be substituted for the other without loss of truth.' What Frege understands by this (since, taken literally, it involves use/mention confusion) is what is often called *Leibniz's Law* – interpreted as comprising both the Principle of the Indiscernibility of Identicals (reading the equivalence from left to right) and the Principle of the Identity of Indiscernibles (reading the equivalence from right to left):

$$x = y \leftrightarrow (\forall F)(Fx \leftrightarrow Fy).$$

As this formulation in modern notation shows, what is provided here is a definition of identity in purely logical terms (allowing quantification over properties); and it is this that supports Frege in taking the concept of identity as already known.

[26] The impossibility of translating 'gleich' everywhere by either 'equal' or 'identical' is shown up here. It is 'equal' more than 'identical' that might be taken to express agreement only in this or that respect; yet whilst we may talk of two lines being *equal* in length, we talk of two surfaces being *identical* in colour. But since Frege wants to show that 'equal' and 'the same' (viz. 'identical') can be treated as synonymous, the alternation in the translation here only highlights Frege's point.

if line *a* is parallel to line *b*. This is made simpler by initially knowing no other assertion about the direction of a line than that it agrees with the direction of another line. We would therefore need to demonstrate only the substitutability in an equality of this kind, or in contents that contain such equalities as components.[Q] All other assertions about directions would first have to be defined, and for these definitions we could adopt the rule that the substitutability of the direction of a line by that of one parallel to it must remain valid.

§66. But yet a third doubt arises about our suggested definition. In the proposition

'The direction of *a* is equal to the direction of *b*'

the direction of *a* appears as an object[R] and we have in our definition a means of reidentifying this object should it appear in another guise,
78 say, as the direction of *b*. But this means | does not provide for all cases. It cannot, for example, be used to decide whether England is the same as the direction of the Earth's axis. Excuse the apparently nonsensical example! Of course, no one is going to confuse England with the direction of the Earth's axis; but that is no thanks to our definition. That says nothing as to whether the proposition

'The direction of *a* is equal to *q*'

is to be affirmed or denied, unless *q* itself is given in the form 'the direction of *b*'. What we lack is the concept of direction; for if we had this, then we could stipulate that if *q* is not a direction, then our proposition is to be denied, and if *q* *is* a direction, then the original definition decides the matter. Now it is natural to offer the definition:

q is a direction, if there is a line *b* whose direction is *q*.

But it is now clear that we have come round in a circle. In order to apply this definition, we would already have to know in each case whether the proposition

[Q] In a hypothetical judgement, for example, an equality of directions could occur as either antecedent or consequent.

[R] The definite article indicates this. A concept is for me a possible predicate of a singular judgeable content, an object a possible subject of such a content. If in the proposition

'The direction of the axis of the telescope is equal to the direction of the Earth's axis'

we take the direction of the axis of the telescope as subject, then the predicate is 'equal to the direction of the Earth's axis'. This is a concept. But the direction of the Earth's axis is only a part of the predicate; it is an object, since it can also be made the subject.

'q is equal to the direction of b'

is to be affirmed or denied.

§67. If one were to say: q is a direction if it is introduced by means of the definition offered above, then the way in which the object q is introduced would be treated as a property of it, which it is not. The definition of an object asserts, as such, really nothing about it, but instead stipulates the meaning [*Bedeutung*] of a symbol. After this has been done, it transforms itself into a judgement which does deal with the object, but now it no longer introduces it but stands on the same level as other assertions about it. If this way out were chosen, it would presuppose that an object can only be given in one single way; for otherwise it would not follow, from the fact that q was not introduced
79 by means of our definition, | that it could not have been so introduced. All equations would then come down to this, that whatever is given to us in the same way is to be recognized as the same. But this is so self-evident and so unfruitful that it is not worth stating. Indeed, no conclusion could ever be drawn here that was different from any of the premises. The multitude of meaningful [*bedeutsame*] uses of equations depends rather on the fact that something can be reidentified even though it is given in a different way.

§68. Since we cannot in this way obtain a sharply defined concept of direction nor, for the same reasons, such a concept of Number, let us try another way. If line a is parallel to line b, then the extension of the concept 'line parallel to line a' is equal to the extension of the concept 'line parallel to line b'; and conversely, if the extensions of these two concepts are equal, then a is parallel to b. Let us therefore suggest the definitions:

the direction of line a is the extension of the concept 'parallel to line a';
the shape of triangle d is the extension of the concept 'similar to triangle d'.

If we want to apply this to our own case, then we have to substitute for directions or triangles concepts, and for parallelism or similarity the possibility of correlating one-one the objects that fall under the one concept with those that fall under the other. If this possibility obtains, I shall speak, for short, of the concept F being *equinumerous*[27] to the concept G, but I must ask that this word be regarded as an arbitrarily chosen form of expression, whose meaning is to be gleaned not from its linguistic construction but from this stipulation.

[27] The German term is 'gleichzahlig', which Austin misleadingly translated as 'equal'. Since the German word was itself an invented one, 'equinumerous' seems an appropriate translation.

I therefore offer the definition:

80 the Number that belongs to the concept F is | the extension[S] of the concept 'equinumerous to the concept F'.

§69. That this definition is correct will hardly, perhaps, be clear at first. For is an extension of a concept not thought to be something different [from a number]? What it is thought to be is evident from the basic assertions that can be made about extensions of concepts. They are the following:

(1) [that] equality [holds between them],
(2) that one is more inclusive [*umfassender*] than another.

Now the proposition

'The extension of the concept "equinumerous to the concept F" is equal to the extension of the concept "equinumerous to the concept G"'

is true if and only if the proposition

'The same number belongs to the concept F as to the concept G'

is also true. Here there is thus complete agreement.

Certainly, we do not say that one number is more inclusive than another, in the sense in which the extension of a concept may be more inclusive than that of another; but the case in which

the extension of the concept 'equinumerous to the concept F'

is more inclusive than |

81 the extension of the concept 'equinumerous to the concept G'

[S] I believe that for 'extension of the concept', simply 'concept' could be said. But two different objections would arise:

1. [that] this contradicts my earlier claim that the individual numbers are objects, as indicated by the definite article in such expressions as 'the number two' and by the impossibility of speaking of ones, twos, etc. in the plural, as well as by the fact that the number constitutes only a part of the predicate of a number statement;
2. that concepts can be of equal extension, without coinciding.

Now I am actually of the opinion that both objections can be met; but that would lead us too far away here. I assume that it is known what the extension of a concept is.

cannot occur either; but rather, if all concepts that are equinumerous to G are also equinumerous to F, then conversely, all concepts that are equinumerous to F are also equinumerous to G. This 'more inclusive' should not, of course, be confused with 'greater', which occurs amongst numbers.

Admittedly, the case can still be imagined in which the extension of the concept 'equinumerous to the concept F' is more inclusive or less inclusive than the extension of another concept, which, according to our definition, could not then be a Number; and it is not usual to call a Number more inclusive or less inclusive than the extension of a concept; but there is also nothing to stop us adopting such a form of speech, should such a case occur.

[The rest of Part IV (§§70–86; *GL*, pp. 81–99), in which Frege provides a sketch of his logicist reduction of arithmetic, is omitted here; but a summary of the argument, with some clarificatory interpolations (in square brackets), is offered below.

Completion of our definition and proof of its worth (§§70–83)

Definitions, Frege writes, prove themselves by their fruitfulness; and his definition of number is to be justified by showing how the well-known properties of numbers can be derived from it (§70). [Although Frege does not himself expressly do so, it is worth noting that from Frege's explicit definition (as given at the end of §68) we can now *derive* the proposition (Nb) that, according to Frege, had been inadequately defined contextually by means of (Na):[28]

(Na) The concept F is equinumerous to the concept G. (There are as many objects falling under concept F as under concept G, i.e. there are just as many F's as G's.)

(Nb) The number of F's is equal to the number of G's. (The number that belongs to the concept F is the same as the number that belongs to the concept G.)

For what we have are the following two explicit definitions:

(Ne) The Number that belongs to the concept F is the extension of the concept 'equinumerous to the concept F'.

(Nε) The Number that belongs to the concept G is the extension of the concept 'equinumerous to the concept G'.

[28] The labelling that follows – (Na), (Nb), etc. – has been added for ease of presentation. For further discussion of (Na) and (Nb), see the Introduction, pp. 15ff. above.

Furthermore, according to Frege (cf. §68), from (Na) we can infer (Nd):

(Nd) The extension of the concept 'equinumerous to the concept F' is equal to the extension of the concept 'equinumerous to the concept G'.

(Nb) clearly then follows from (Nd), (Ne) and (Nε). What we have thus done is derive (Nb) not *directly* from (Na), but *indirectly* via (Nd) and the explicit definitions. So if – *pace* Frege himself – we felt unhappy about the explicit definitions, but found the contextual method legitimate, we could still accept Frege's starting-point, the move from (Na) to (Nb).[29]]

The first step is to provide a more exact definition of 'equinumerosity' [involved in both (Na) and (Ne)]. Frege has already indicated that this is to be defined in terms of one-one correlation (cf. §§63, 68), and the key point here is that this can itself be characterized independently of number (despite the phrase '*one-one* correlation'). Frege gives an example to illustrate the idea: 'If a waiter wants to be sure of laying just as many knives as plates on a table, he does not need to count either of them, if he simply lays a knife right next to each plate, so that every knife on the table is located right next to a plate. The plates and knives are thus correlated one-one, by means of the same spatial relationship.' (§70.) Generalizing, then, two concepts F and G are equinumerous if there is a relation R that correlates one-one the objects falling under F with the objects falling under G, and this, as Frege had already shown in the *Begriffsschrift*, can be characterized purely logically. [In modern notation, 'Rxy' symbolizing that x stands in relation R to y, this can be formalized as follows:

$$\text{(Na*)}\quad (\forall x)(Fx \rightarrow (\exists y)[Gy \;\&\; (\forall z)(Rxz \leftrightarrow z = y)\,]) \\ \&\; (\forall y)(Gy \rightarrow (\exists x)[Fx \;\&\; (\forall w)(Rwy \leftrightarrow w = x)\,]).$$

The first conjunct says that for any F (i.e. anything that is an F), there is one and only one G to which it is R-related, and the second conjunct adds that for any G, there is one and only one F to which it is R-related. (The first clause, in other words, states the condition for the relation between the F's and the G's to be *many-one*, and the second clause the condition for the relation to be *one-many*, the two clauses providing the combined condition for the relation to be *one-one* – cf. (OO)

[29] This is one of the central insights that motivates Wright (1983), in his reconstruction of Frege's arguments. As Dummett (1991a: p. 123) notes, Frege does, in fact, himself derive all his theorems from the original contextual equivalence without further appeal to his explicit definition.

on p. 77 above.)] As Frege remarks in §72, in offering the same analysis there, this 'reduces one-one correlation to purely logical relationships'.

Returning to the problem that Frege felt had been unresolved in §56, we do now have a way of determining whether the number that belongs to the concept F is the same as the number that belongs to the concept G. [Frege's definitions of propositions of the form 'The number n belongs to the concept F' ('There are n F's') were regarded by him as unsatisfactory (§56), because they did not adequately determine the relevant objects. But propositions of this form, according to Frege (cf. §57), are reducible to propositions that have the preferred form of an equation (identity statement):

(NF) The number n is the Number that belongs to the concept F.]

The expression 'n is a Number' is taken as equivalent [*gleichbedeutend*] to the expression 'there is a concept such that n is the Number that belongs to it'; and this is now seen as acceptable with Frege's explicit definition [(Ne)] in place. 'Thus the concept of Number is defined, admittedly, it seems, in terms of itself, but nevertheless without error, since "the Number that belongs to the concept F" is already defined [as "the extension of the concept 'equinumerous to the concept F'"].' (§72.)

All that is then needed to provide definitions of the individual numbers is to find appropriate concepts [to substitute in (NF)]. In the case of the number 0, Frege utilizes the concept *not identical with itself*, yielding the following definition (cf. §74):

[(N0)] The number 0 is the Number which belongs to the concept *not identical with itself* [*sich selbst ungleich*].

In offering this, Frege remarks that there is no objection to taking a concept that contains a contradiction, so long as we do not assume that something falls under it: 'All that can be demanded of a concept on the part of logic and for rigour of proof is its sharp boundary, that for every object it is determined whether it falls under the concept or not. Now this demand is completely satisfied by a concept containing a contradiction such as "not identical with itself"; since of every object it is known that it does not fall under such a concept.' (§74.) Furthermore, the crucial point about Frege's chosen concept is that it can be specified purely logically ('$x \neq x$'), utilizing the Leibnizian definition of identity given in §65. [From (Ne) and (N0) we can then formulate an explicit definition that satisfies Frege's requirements:

(E0) The number 0 is the extension of the concept 'equinumerous to the concept *not identical with itself*'.

Assuming, with Frege, that the notion of an extension is unproblematically a logical notion,[30] we have indeed then managed to characterize the number 0 in purely logical terms.]

The next step in the project is to define the successor relation, relating any two adjacent members of the natural number series. Frege offers this definition of 'n follows in the natural number series immediately after m' (§76):

[(SR)] There is a concept F, and an object x falling under it, such that the Number that belongs to the concept F is n and the Number that belongs to the concept *falling under F but not identical with x* is m.

Intuitively, this clearly gives the desired result: there is one less object falling under the latter concept than under the former, and the relationship between the two concepts can be characterized purely logically [cf. (F_{n+1}) in fn. 20, p. 105 above].

Frege goes on to show how the definition yields 1 as the successor of 0 (§77). Take the concept *identical with 0*. Since one and only one object falls under this concept, namely, the number 0, the Number that belongs to this concept is the number 1. The Number that belongs to the concept *falling under the concept 'identical with 0' but not identical with 0*, on the other hand, is clearly 0, since nothing can fall under this concept. So the condition stated in (SR) is satisfied (taking 'F' as 'identical with 0', giving $x = 0$, $n = 1$ and $m = 0$), and we can conclude that 1 is the successor of 0. What Frege has done here, in other words, is provide a suitable concept to substitute in (NF) to generate a definition of the number 1:

[(N1)] The number 1 is the Number that belongs to the concept *identical with 0*.

Since 0 has already been defined purely logically, and in fact is the only object that has been so defined up to this point, the concept *identical with 0* is obviously the ideal concept for Frege to take in order to define the number 1 logically. What the argument just given then shows is that this is indeed the number that follows in the natural number series immediately after 0. (Cf. §77.)

[From (Ne) and (N1), the following explicit definition can then be offered:

(E1) The number 1 is the extension of the concept 'equinumerous to the concept *identical with 0*'.

[30] Cf. §68, fn. S (p. 115 above), and §107 (p. 128 below).

With the numbers 0 and 1 now defined, the number 2 can then be generated in a similar way:

(N2) The number 2 is the Number that belongs to the concept *identical with 0 or 1*.
(E2) The number 2 is the extension of the concept 'equinumerous to the concept *identical with 0 or 1*'.

The pattern that emerges is clear: each number can be defined in terms of its predecessor(s), since the natural number series up to a given number n has itself $n + 1$ members (since it starts from 0). This suggests the following general definition (cf. §79):

(Nn+1) The number $n + 1$ is the Number which belongs to the concept *member of the natural number series ending with n*.

Of course, the concept *member of the natural number series ending with n* itself needs to be defined, but once again, the materials for doing so had already been supplied in the *Begriffsschrift* (§§26–9; see pp. 75–6 above), where a logical characterization had been offered, through the notion of an hereditary property, of '*b* follows *a* in the *f*-series' (cf. *GL*, §79), from which '*b* is a member of the *f*-series beginning with *a*' could then be defined. Since this is equivalent to '*a* is a member of the *f*-series ending with *b*', the required logical definition can be provided (cf. *GL*, §81). (SR) can then be used to show that (Nn+1) yields $n + 1$ as the successor of n – substituting 'member of the natural number series ending with n' for 'F', 'n' for 'x', '$n + 1$' for 'n', and 'n' for 'm' (cf. *GL*, §§82–3).[31]]

With Frege's definitions in place, it becomes possible to derive the familiar properties of the natural numbers. For example, [(Nn+1)] implies that every natural number has a successor, i.e. that no member of the natural number series follows after itself, as Frege puts it in §83. In the *Grundlagen* Frege merely states a handful of theorems (§78); the full task was to be undertaken in the *Grundgesetze*.[32]

Infinite Numbers (§§84–86)

In the final subdivision of Part IV, Frege makes some brief remarks about infinite (transfinite) Numbers, the existence of which is unproblematic

[31] Frege provides only a sketch here; a fuller proof is given in *GG*, I, §§114–19.
[32] The formal proofs are presented in Part II of *GG* (Vol. I, §§53–179; Vol. II, §§1–54), which has not as yet been translated into English (not that there is much to translate: the vast majority of it is written in Frege's symbolic notation). A useful summary of the main theorems derived in Part II, however, is provided in Currie, 1982: pp. 55–7.

on his account of number. For the Number that belongs to the concept *finite Number*, defined as the concept *member of the natural number series beginning with 0* (§83) is clearly an infinite Number, which Frege symbolizes by '∞_1' ('$\aleph_0$', as it is now written). 'There is nothing at all weird or wonderful about the infinite Number ∞_1 so defined. "The Number that belongs to the concept F is ∞_1" means [*heisst*] no more nor less than: there is a relation that correlates one-one the objects falling under the concept F with the finite Numbers. According to our definitions, this has a perfectly clear and unambiguous sense; and that is sufficient to justify the use of the symbol ∞_1 and secure it a meaning [*Bedeutung*]. That we can form no idea of an infinite Number is quite irrelevant and applies just as much to finite Numbers. Our Number ∞_1 is in this way just as definite as any finite Number: it can without doubt be recognized as the same again and be distinguished from another.' (§84.) Frege goes on to express his agreement with Cantor that infinite Numbers are as legitimate as finite Numbers (§85), though he does suggest that his own method of introducing infinite Numbers, through logical definition, is superior to Cantor's appeal to 'inner intuition' (§86). Furthermore, Frege notes, since on his account numbers are characterized right from the start as belonging to concepts, there is no extension of the meaning of 'Number' when infinite numbers are introduced (since they too are attached to concepts), so that worries about invalidating any fundamental laws are minimized (§85).

The translation resumes at the beginning of the concluding part.]

99

V. Conclusion

§87. I hope in this work to have made it probable that arithmetical laws are analytic judgements and therefore *a priori*. Accordingly, arithmetic would be simply a further developed logic, every arithmetical theorem a logical law, albeit a derivative one. Applications of arithmetic in natural science would be logical processing of observed facts;[T] calculation would be inference. The laws of number will not need, as Baumann[U] thinks, to prove their worth in practice in order to be applicable to the external world; for in the external world, in the totality of the spatial, there are no concepts, no properties of concepts, no numbers. The laws of number are thus not really applicable to external things: they are not laws of nature. But they are certainly applicable to judgements that are made about things in the external world: they are laws of the laws of nature.

[T] Observation itself already involves logical activity.
[U] Op. cit., Vol. II, p. 670.

They do not assert a connection between natural phenomena, but a connection between judgements; and the latter include the laws of nature.

§88. Kant[V] obviously underestimated the value of analytic judgements – no doubt as a result of defining the concept too narrowly, although
100 the broader concept used here | does appear to have been in his mind.[W] On the basis of his definition, the division into analytic and synthetic judgements is not exhaustive. He is thinking of the case of the universal affimative judgement. Here one can speak of a subject-concept and ask – according to the definition – whether the predicate-concept is contained in it. But what if the subject is an individual object? What if the question concerns an existential judgement? Here there can be no talk at all of a subject-concept in Kant's sense. Kant seems to think of a concept as defined by a conjunction of marks;[33] but this is one of the least fruitful ways of forming concepts. Looking back over the definitions given above, there is scarcely one of this kind to be found. The same holds too of the really fruitful definitions in mathematics, for example, of the continuity of a function. We do not have here a series of conjunctions of marks, but rather a more intimate, I would say more organic, connection of defining elements. The distinction can be clarified by means of a geometrical analogy. If the concepts (or their extensions) are represented by areas on a plane, then the concept defined by a conjunction of marks corresponds to the area that is common to all the areas representing the marks; it is enclosed by sections of their boundaries. With such a definition it is thus a matter – in terms of the analogy – of using the lines already given to demarcate an area in a new way.[X] But nothing essentially new comes out of this. The more fruitful definitions of concepts draw boundary lines that were not there at all.
101 What can be inferred from them cannot be seen from the start; | what was put into the box is not simply being taken out again. These inferences extend our knowledge, and should therefore be taken as synthetic, according to Kant; yet they can be proved purely logically and are thus analytic. They are, in fact, contained in the definitions, but like a plant in a seed, not like a beam in a house. Often several definitions are needed for the proof of a proposition, which is not therefore contained in any single one and yet does follow purely logically from all of them together.

[V] Op. cit., Vol. III, pp. 39ff. [Cf. Kant, *Critique of Pure Reason,* A6ff./B10ff.]

[W] On p. 43 [B14] he says that a synthetic proposition can only be recognized by the law of contradiction, if another synthetic proposition is presupposed. [Cf. Frege's fn. E to §3, p. 92 above.]

[X] Similarly, if the marks are connected by 'or'.

[33] E.g. defining 'horse' as 'four-footed, solid-hoofed and herbivorous mammal'. For the notion of a 'mark' ('Merkmal'), see §53 (pp. 102–3 above); *CO*, pp. 189–90 below.

§89. I must also contradict the generality of Kant's[Y] claim that without sensibility no object would be given to us. Zero and one are objects that cannot be given to us through the senses. Even those who regard the smaller numbers as intuitable will surely have to concede that none of the numbers greater than $1000^{1000^{1000}}$ can be given to them in intuition, and yet we know various things about them. Perhaps Kant used the word 'object' in some other sense; but then zero, one and our ∞_1 entirely drop out of his account; for they are not concepts either, and even of concepts Kant[Y] requires that objects be associated with them in intuition.

In order not to lay myself open to the charge of simply picking holes in the work of a genius to whom we can only look up with grateful admiration, I think I should also emphasize the agreement that by far prevails. To touch only on what is salient here, I see Kant as having performed a great service in drawing the distinction between synthetic and analytic judgements. In calling geometrical truths synthetic and *a*
102 *priori*, he revealed their true | nature. And this is still worth repeating now, since it is still not often recognized. If Kant was wrong about arithmetic, then that does not, I believe, detract fundamentally from the service he performed. What mattered to him was the existence of synthetic *a priori* judgements; whether they occur only in geometry or also in arithmetic is of less significance [*Bedeutung*].

§90. I do not claim to have made the analytic nature of arithmetical propositions more than probable, since it can still always be doubted whether their proof can be completely constructed from purely logical laws, or whether an assumption of another kind has not intruded somewhere unnoticed. Nor will this doubt be fully allayed by the indications I have given of the proof of some propositions; it can only be removed by a chain of inference free of gaps, with no step taken that is not in accord with one of a few modes of inference recognized as purely logical. Until now hardly a proof has been constructed like this, since the mathematician is content if every transition to a new judgement is self-evidently correct, without enquiring into the nature of this self-evidence, whether it is logical or intuitive. Such a transition is often very complex and equivalent to several simple inferences, alongside which something from intuition can still enter. Progress is by leaps, and from this arises the apparently abundant variety of modes of inference in mathematics; for the bigger the leaps, the more complex the combinations of simple inferences and intuitive axioms they can represent. Nevertheless, such a transition is often immediately self-evident to us, without our being aware of the intermediate steps, and since it does not present

[Y] Op. cit., Vol. III, p. 82 [*Critique of Pure Reason,* A51/B75].

itself as one of the recognized logical modes of inference, we are all too ready to take this self-evidence as intuitive and the inferred truth as
103 synthetic, even | when its domain of validity obviously extends beyond the intuitable.

It is not possible this way to separate cleanly the synthetic that is based on intuition from the analytic. Nor is it possible to draw up with certainty a complete list of axioms of intuition, from which every mathematical proof can be constructed according to logical laws.

§91. The requirement that all leaps in an argument be avoided cannot therefore be repudiated. That it is so hard to satisfy lies in the prolixity of a step by step approach. Every proof that is only slightly complicated threatens to become monstrously long. In addition, the enormous variety of logical forms revealed in ordinary language makes it difficult to delimit a set of modes of inference that covers all cases and is easy to survey.

To reduce these deficiencies, I devised my *Begriffsschrift*. It is intended to achieve greater economy and surveyability of expression and to be used in a few fixed forms in the manner of a calculus, so that no transition is permitted that is not in accord with the rules that are laid down once and for all.[Z] No assumption can then slip in unnoticed. In this way I have proved, without borrowing an axiom from intuition, a theorem[AA] that might at first sight be taken as synthetic, which I shall here formulate thus:

> If the relation of every member of a series to its successor is many-one [*eindeutig*], and if *m* and *y* follow *x* in this series, then either *y* precedes *m* in this series or coincides with *m* or follows *m*. |

104 From this proof it can be seen that propositions that extend our knowledge can contain analytic judgements.[BB]

[The remaining sections of the book fall under the heading 'Other numbers', and the first twelve sections (§§92–103; *GL*, pp. 104–13), in which Frege is mainly concerned to refute what he calls the formalist theory [*formale Theorie*], are here omitted. The formalist is understood as someone who imagines that one need only postulate that, say, the laws of addition and multiplication, as defined over the natural numbers, hold for any extension of the number system, in order to investigate coherently the properties of that extended system (cf. §96).[34] But, Frege argues, it is quite wrong to suppose that a concept has instances if no contradiction has yet revealed itself – not only are self-contradictory concepts admissible, but even if a concept contains no contradiction, that is still no guarantee that anything falls under it (cf. §§94, 96): 'even the mathematician cannot create whatever he likes, any more than the geographer; he too can only discover what is there and name it' (§96). Frege remarks that 'It is common to act as if mere postulation [*Forderung*] were already its own fulfilment' (§102).[35] Yet 'postulating', say, that through any three points a straight line can be drawn is simply incoherent; and we first have to prove that our postulates contain no contradiction (cf. §102). With the introduction of new numbers, Frege writes, 'the meaning [*Bedeutung*] of the words "sum" and "product" is extended' (§100), and we cannot automatically assume that initial definitions of basic concepts remain valid in any enlarged system (cf. §102).[36] But if we cannot just define new numbers into existence by specifying a list of properties that characterize them, nor arrive at them by simply extending an existing number system taking its axioms for granted, how are they then to be apprehended? Frege takes up this question in §104.]

114 **§104.** How, then, are fractions, irrational numbers and complex numbers to be given to us? If we turn for help to intuition, then we introduce something foreign into arithmetic; but if we only define the concept of such a number by its marks, if we only require that the number have certain properties, then nothing guarantees that anything falls under the concept and corresponds to our demands, and yet it is precisely on this that proofs must rest.

Now how is it in the case of the [natural] Numbers? Should we really not talk of $1000^{1000^{1000}}$ before that many objects have been given to us in intuition? Is it until then an empty symbol? No! It has a quite definite sense, even though it is psychologically impossible, in view of the

[Z] It is intended, however, to provide a means of expressing not only logical form, like Boolean symbolism, but also content.

[AA] *Begriffsschrift*, p. 86, formula 133.

[BB] This proof will still be found far too lengthy, a disadvantage that might seem to more than outweigh the near absolute certainty of a mistake or a gap [in a shorter proof]. My aim at the time was to reduce everything to the smallest possible number of the simplest possible logical laws. As a result, I used only a single mode of inference. But even then I pointed out in the Preface, p. vii [p. 51 above], that for further applications more modes of inference would be recommended. This can be done without affecting the validity of the chain of inference, and thus significant [*bedeutende*] abbreviation can be achieved.

[34] The formalism here is what Dummett (1991a: p. 178) has suggested should be better called 'postulationism', to distinguish it from more sophisticated forms, both those that Frege later attacks in *GG*, II, §§86–137, and those that have subsequently been developed.

[35] Cf. Russell's famous comment that 'The method of "postulating" what we want has many advantages; they are the same as the advantages of theft over honest toil' (*IMP*, p. 71).

[36] Cf. *GG*, II, §§56–65 (pp. 259–68 below), where Frege objects very strongly to what he calls in §57 'the mathematicians' favourite procedure, piecemeal definition'.

brevity of our life, for us to apprehend so many objects;[CC] but nevertheless $1000^{1000^{1000}}$ is an object, whose properties we can recognize, even though it is not intuitable. We can convince ourselves of this by showing that one and only one positive whole number is always expressed by a^n, the symbol introduced for the nth power of a, where a and n are positive whole numbers. To explain this in detail here would lead us too far away. The general strategy will be clear from the way we defined zero in §74, one in §77, and the infinite Number ∞_1 in §84, and from the sketch of the proof that every finite Number in the natural number series has a successor (§§82–83).

So too in the case of the definitions of fractions, complex numbers, etc., everything will depend in the end on finding a judgeable content that can be transformed into an equation whose sides are precisely the
115 new | numbers. In other words, we must fix the sense of a recognition judgement [*Wiedererkennungsurteil*] for such numbers. In doing so, we must heed the doubts that we discussed, in §§63–68, concerning such a transformation. If we proceed in the same way as we did there, then the new numbers will be given to us as extensions of concepts.

§105. On this conception of numbers,[DD] it seems to me, the attraction that work on arithmetic and analysis holds is easily explained. Adapting the familiar words, it might well be said: the real object of reason is reason itself.[37] We are concerned in arithmetic not with objects that become known to us through the medium of the senses as something foreign from outside, but with objects that are immediately given to reason, which can fully comprehend them, as its own.[EE]

And yet, or rather precisely because of this, these objects are not subjective fantasies. There is nothing more objective than arithmetical laws.

[CC] A rough estimate shows that millions of years would not suffice for this.

[DD] If too might be called formalist [*formal*]. Yet it is quite different from what was criticized above under this name.

[EE] By this I do not in the least want to deny that without sense impressions we are as thick as a plank and know nothing of numbers or of anything else; but this psychological proposition does not concern us here at all. I emphasize this again because of the constant danger of confusing two fundamentally different questions.

[37] 'der eigentliche Gegenstand der Vernunft ist die Vernunft'. This is presumably an allusion to a remark in Ottilie's Journal in Goethe's novel *Die Wahlverwandtschaften* (Part Two, Chapter Seven): 'das eigentliche Studium der Menschheit ist der Mensch' ('the proper study of mankind is man'); though this remark too has historical predecessors, for example, in Pierre Charron's *De la sagesse* (Bordeaux, 1601, Book I, chapter I, p. 1): 'La vraye science & le vray estude de l'homme, c'est l'homme' – cf. Thiel, 1986: p. 172 – and in Pope's *Essay on Man* (Epistle II, lines 1–2): 'Know then thyself, presume not God to scan, | The proper study of mankind is man'. Cf. also Kant, *Critique of Pure Reason*, preface to the first edition, where Kant writes that 'I have to deal simply with reason itself and its pure thinking' ('daß . . . ich es lediglich mit der Vernunft selbst und ihrem reinen Denken zu tun habe'; Axiv).

§106. Let us now cast a brief glance back over the course of our investigation. After establishing that number is neither a collection of things nor a property of such, nor a subjective product of mental processes, but rather, that a statement of number asserts something objective about a concept, we first attempted to define the individual numbers 0, 1, etc., and the relation of succession in the number series. The first attempt failed, because we had only defined each assertion about |
116 concepts, but not 0, 1 separately, which are only parts of [the predicate involved in] the assertion. This had the result that we were unable to prove the equality of numbers. It showed that the numbers with which arithmetic is concerned must be grasped not as dependent attributes but substantivally.[FF] Numbers thus appeared as reidentifiable objects, though not as physical or even merely spatial ones, nor as ones which we can picture through the power of imagination. We then laid down the principle that the meaning of a word is to be defined not in isolation, but in the context of a proposition; only by adhering to this, I believe, can the physical conception of number be avoided, without falling into a psychological one. Now there is one kind of proposition that, for every object, must have a sense, that is, recognition statements, called equations in the case of numbers. As we saw, statements of number too are to be construed as equations. It thus came down to fixing the sense of a numerical equation, expressing it without making use of number words or the word 'number'. The possibility of correlating one-one the objects falling under concept F with those falling under concept G we recognized as the content of a recognition judgement concerning numbers. Our definition thus had to lay it down that this possibility means the same as a numerical equation. We recalled similar cases: the definition of direction in terms of parallelism, shape in terms of similarity, etc.

§107. The question then arose: when is it justified to construe a content as that of a recognition judgement? For this the condition must be
117 fulfilled | that in every judgement the left-hand side of the putative equation can be substituted by the right-hand side without altering its truth. Now, without adding further definitions, we do not initially know anything about the left- or right-hand sides of such an equation than just that they are equal. So all that needed to be demonstrated was substitutability in an equation.

But there still remained one doubt. A recognition statement must always have a sense. If we now construe the possibility of correlating one-one the objects falling under concept F with those falling under concept G as an equation, by saying: 'the Number that belongs to the concept F is equal to the Number that belongs to the concept G',

[FF] The distinction corresponds to that between 'blue' and 'the colour of the sky'.

hereby introducing the expression 'the Number that belongs to the concept F', then the equation only has a sense if both sides have this same form. According to such a definition, we could not judge whether an equation is true or false, if only one side has this form. This led us to the definition:

> The Number that belongs to the concept F is the extension of the concept 'concept equinumerous to the concept F', where a concept F is called equinumerous to a concept G if the possibility exists of one-one correlation.

We assumed here that the sense of the expression 'extension of a concept' was known. This way of overcoming the difficulty may well not meet with universal approval, and many will prefer removing the doubt in another way. I too attach no great importance to the introduction of extensions of concepts.

§108. It now still remained to define one-one correlation; we reduced
118 this to purely | logical relations. After we had then indicated the proof of the proposition 'The number that belongs to the concept F is equal to the number that belongs to the concept G, if the concept F is equinumerous to the concept G', we defined 0, the expression 'n directly follows m in the natural number series', and the number 1, and showed that 1 directly follows 0 in the natural number series. We cited a few theorems, which can easily be proved at this point, and then went a little more deeply into the following proposition, which reveals the infinity of the number series:

> 'Every number in the natural number series has a successor'.

We were thus led to the concept 'member of the natural number series ending with n', from which we could show that the Number belonging to this directly follows n in the natural number series. We first defined it by means of the general relation of the following in a ϕ-series of an object x by an object y. The sense of this expression too was reduced to purely logical relations. And this enabled us to prove that the inference from n to $(n + 1)$, which is usually regarded as specifically mathematical, is based on general logical modes of inference.

To prove the infinity of the number series, we then needed the theorem that no finite number follows in the natural number series after itself. We thus arrived at the concepts of finite and infinite number. We showed that the latter is fundamentally no less logically justified than the former. By way of comparison, Cantor's infinite Numbers and his 'following in a succession' were considered, and the difference in formulation pointed out.

§109. From all that has gone before, the analytic and *a priori* nature of arithmetical truths has thus emerged as highly probable; and we
119 achieved | an improvement on Kant's view. We further saw what is still missing in order to raise this probability to certainty, and indicated the path that must lead to this.

Finally, we used our results in a critique of a formalist theory of negative, fractional, irrational and complex numbers, which showed up its inadequacies. We recognized its error in assuming as proved that a concept is free from contradiction if no contradiction has revealed itself, and in taking freedom from contradiction as sufficient guarantee that something falls under the concept. This theory imagines that it need only formulate postulates, whose fulfilment then takes care of itself. It behaves like a god, who can create by his mere word whatever he needs. It must also be reprimanded for passing off as a definition what is only a set of instructions, the following of which would introduce something foreign into arithmetic; even though in its formulation it might be regarded as innocent, this is only because it remains a mere set of instructions.

This formalist theory is thus in danger of lapsing back into an *a posteriori* or at least synthetic theory, however much it may give the appearance of soaring on the heights of abstraction.

Now our earlier account of the positive whole numbers shows us the possibility of avoiding the confusion with external things and geometrical intuitions, yet without making the mistake of the formalist theory. As there, it depends on fixing the content of a recognition judgement. If we think of this as everywhere achieved, then negative, fractional, irrational and complex numbers appear as no more mysterious than the positive whole numbers, which are no more real, actual or tangible than they.